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ABSTRACT

This work focuses on the construction of equilibrated two-phase antiplane shear deformations of a *non-elliptic isotropic* and *incompressible hyperelastic* material. It is shown that this material can sustain *metastable* two-phase equilibria which are neither piecewise homogeneous nor axisymmetric, but, rather, involve non-planar interfaces which completely segregate inhomogeneously deformed material in distinct elliptic phases. These results are obtained by studying a constrained boundary value problem involving an interface across which the deformation gradient jumps. The boundary value problem is recast as an integral equation and conditions on the interface sufficient to guarantee the existence of a solution to this equation are obtained. The contraints, which enforce the segregation of material in the two elliptic phases, are then studied. Sufficient conditions for their satisfaction are also secured. These involve additional restrictions on the interface across which the deformation gradient jumps—which, with all restrictions satisfied, constitutes a *phase boundary*. An uncountably infinite number of such phase boundaries are shown to exist. It is demonstrated that, for each of these, there exists a solution—unique up to an additive constant—for the constrained boundary value problem. As an illustration, approximate solutions which correspond to a particular class of phase boundaries are then constructed. Finally, the kinetics and stability of an arbitrary element within this class of phase boundaries are analyzed in the context of a quasistatic motion.

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1. INTRODUCTION

Finite elastic equilibria with discontinuous deformation gradients have figured prominently in recent continuum mechanical treatments of displacive solid-solid phase transformations. Models of this sort are pertinent to the investigation of *shape memory*, *twinning* and *transformation toughening* in solids. Shape memory and transformation toughening occur in metallic and ceramic alloys, respectively, while twinning can be found in both metallic and ceramic alloys. Micrographs of multiphase equilibrium states in alloys, such as those presented by ZACKAY, JUSTUSSON, & SCHMATZ [27] and PORTER & HEUER [21], often display configurations wherein the various phases are segregated by geometrically complicated interfaces. One question which arises regarding the aformentioned continuum mechanical idealizations of such materials is whether they are capable of capturing the morphological complexity of such deformations. As a first step toward answering this question, this work focuses, within the context of a particular class of hypothetical materials, on the construction of equilibria involving coexistent phases segregated by surfaces which—although not as morphologically complex as those displayed in [21,27]—are, at least, non-planar.

In a *homogeneous*, *hyperelastic* material discontinuous deformation gradients occur only if the relevant *elastic potential* allows for a loss of ellipticity, at certain values of the deformation gradient, in the associated displacement equations of equilibrium.[1] Materials characterized by elastic potentials which allow such a loss of ellipticity are referred to as *non-elliptic*. Of particular importance in this work are non-elliptic materials which have at least two disjoint elliptic phases. Examples of such materials are provided by ERICKSEN [12] in the context of a one-dimensional bar theory, and by ABEYARATNE [3] in his study involving a special class of *incompressible*, *isotropic* materials. ABEYARATNE [3], BALL & JAMES [10], GURTIN [16], and SILLING [25] have demonstrated that materials of this sort support equilibrium states which display coexistent elliptic phases and, in addi-

[1] For a discussion of this issue see, for instance, ROSAKIS [24].

tion, minimize the relevant energy functional. As a result of the latter property these states are referred to as *mechanically stable.* The associated deformation fields in all of the foregoing works are either piecewise homogeneous or axisymmetric. In the case of equilibrated piecewise homogeneous deformations the associated phase boundaries must be planar. BALL & JAMES [10] and SILLING [25] have shown, however, that *energy-minimizing sequences* of piecewise homogeneous mechanically stable deformations may possess limits which are *metastable* as opposed to mechanically stable and, moreover, involve non-planar phase boundaries. On the other hand, ABEYARATNE [2] and SILLING [25] have constructed, respectively, asymptotic and numerical solutions to a boundary value problem involving a mode III crack in a particular subclass of incompressible, isotropic non-elliptic materials. These solutions are not mechanically stable and are neither piecewise homogeneous nor axisymmetric; furthermore, they include the non-elliptic material phase and, in addition, transitions between the elliptic and non-elliptic phases which do not involve jumps in the deformation gradient. These solutions do, however, involve surfaces which separate the two elliptic phases present in the deformation. The relevant interfaces are, moreover, non-planar. ROSAKIS [23] has recently shown that a special class of *anisotropic* non-elliptic materials is capable of sustaining equilibria in which a family of cusped *lenticular* inclusions of one elliptic phase reside in a matrix of another elliptic phase. These states are, in general, metastable.

As yet there are no results pertaining to the existence, in non-elliptic isotropic hyperelastic materials, of multiphase equilibrium states which are free of the non-elliptic phase and are neither piecewise homogeneous nor axisymmetric. The primary objective of this investigation is to prove constructively that a class of non-elliptic isotropic incompressible hyperelastic materials is capable of sustaining deformations of this type. These deformations will typically be metastable—like those associated with the limits of the aformentioned minimizing sequences of piecewise homogeneous deformations and the states constructed by ROSAKIS [23].

Although non-planar phase interfaces may, in reality, reflect anisotropic effects, these results show that they can exist within the context of a model which does not take anisotropy into consideration. Isotropic materials may, consequently, be useful in preliminary studies of the kinetics and stabilty of interfaces between phases. These issues are taken up briefly in the final section of this work and, more thoroughly, by FRIED [13] in a linear stability analysis of states involving planar phase interfaces for a class of non-elliptic isotropic materials.

Chapter 2 is devoted to preliminaries. After a brief overview of the notation to be used, Section 2.1 introduces the kinematics and fundamental balance principles which will be needed in the following. Section 2.2 explains the constitutive restrictions which will be adhered to throughout this work. Section 2.3 begins by introducing the concept of a *quasistatic motion.* It then discusses the notions of mechanical dissipation and driving traction associated with surfaces across which the deformation gradient jumps; these lead naturally to the consideration of a *kinetic relation* and the associated *kinetic response function.* In the final section of Chapter 2, the kinematics are specialized to those of *antiplane shear.*

Chapter 3 focuses upon the solution of a particular constrained boundary value problem, in antiplane shear, involving the field equations and jump conditions put forth in Section 2.4. After formulating the problem in Section 3.1, a representation for the solution of the boundary value problem is presented in Section 3.2. This representation is indeterminate in that it involves the unknown jump in the normal derivative of the displacement field over an interface across which the deformation gradient is discontinuous. In Section 3.3 an integral equation is derived for the unknown jump in the normal derivative of displacement in terms of a parameterization of the interface. Sufficient conditions for the existence of a unique solution of this integral equation are then obtained. These constitute analytical restrictions on the interface geometry. It transpires that there exist an uncountably infinite number of interfaces which comply with these resrictions. It is then shown that for each of these interfaces there exists a solu tion, unique up to an additive constant, tc the boundary value problem stated in Section 3.1. In Section 3.4 the constraints which enforce segregation of the elliptic phases are analyzed. These impose further analytical restrictions on the interface geometry. It is shown that, within the set of interfaces which allow a solution to the boundary value problem, there exist an uncountably infinite number of interfaces which also satisfy these restrictions and, hence, allow a solution to the constrained boundary value problem. Each of these solutions involves a non-planar and non-axisymmetric phase interface which separates elliptic phases subjected to inhomogeneous deformations. Section 3.5 illustrates the results of the two preceeding sections in determining a particular class of non-planar surfaces for which the constrained boundary value problem can be solved. Approximations for the strain and displacement fields corresponding to the solutions of the appropriate family of constrained boundary value problems are then constructed.

The last chapter is concerned with the kinetics and stability of slowly propagating phase boundaries. In Section 4.1 the distribution of driving traction along a phase interface of the kind constructed in Section 3.5 is calculated. Section 4.2 is concerned with observations pertaining to the kinetics of such a surface. Ingredients crucial to this analysis are the kinetic relation and response function introduced in Section 2.3. This section concludes with results that relate the monotonicity of the kinetic response function for a particular material and the *kinetic stability* of that material. These final results are consistent with those obtained in [13].

2. PRELIMINARIES

2.1. Notation, kinematics and balance principles. In the following $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers. The intervals $(0,\infty)$ and $[0,\infty)$ are represented by $\mathbb{R}_+$ and $\bar{\mathbb{R}}_+$. The symbol $\mathbb{R}^n$, with n equal to 2 or 3, represents real n-dimensional space equiped with the standard Euclidean norm. If U is a set, then its closure, interior and boundary are designated by $\bar{U}$, $\mathring{U}$, and ∂U, respectively. The complement of a set V with respect to U is written as $U \setminus V$. Given a function $\psi : U \to W$ and a subset V of U, $\psi(V)$ stands for the image of V under the map ψ.

Vectors and linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ (referred to herein as *tensors*) are distinguished from scalars with the aid of boldface type—lower and upper case for vectors and tensors, respectively. The symbol $\mathcal{L}$ refers to the set of tensors, $\mathcal{L}_+$ denotes the set of all tensors with positive determinant, and $\overset{+}{\mathcal{S}}$ stands for the collection of all symmetric positive definite tensors. The set of unit vectors in $\mathbb{R}^3$ is designated by $\mathcal{N}$. If $\mathbf{F}$ is in $\mathcal{L}$ then $\mathbf{F}^T$ represents its transpose; if, moreover, $\det \mathbf{F} \neq 0$, then the inverse of $\mathbf{F}$ and its transpose are written as $\mathbf{F}^{-1}$ and $\mathbf{F}^{-T}$, respectively. When component notation is used, Greek indices range only over $\{1,2\}$; summation of repeated indices over the appropriate range is implicit.

Let q lie in $[1,\infty)$. Then, a function $\psi : \mathbb{R} \to \mathbb{R}$ is an element of $L^q(\mathbb{R})$ if it is q integrable on $\mathbb{R}$—that is, if its L^q norm over $\mathbb{R}$,

$$\|\psi\|_{L^q(\mathbb{R})} = \Big(\int_{-\infty}^{+\infty} |\psi(\xi)|^q \, d\xi \Big)^{\frac{1}{q}},$$

is defined. Similarly, ψ is an element of $L^\infty(\mathbb{R})$ if ψ is bounded on $\mathbb{R}$—that is, if its L^∞ norm over $\mathbb{R}$,

$$\|\psi\|_{L^\infty(\mathbb{R})} = \sup_{\xi \in \mathbb{R}} |\psi(\xi)|,$$

exists. A function $\Psi : \mathbb{R}^2 \to \mathbb{R}$ is an element of $L^q(\mathbb{R}^2)$ or $L^\infty(\mathbb{R}^2)$, respectively, if the analogous L^q or L^∞ norm over $\mathbb{R}^2$ exists.

Consider, now, a body $\mathcal{B}$ which, in a reference configuration, occupies a region $\mathcal{R}$ contained in $I\!R^3$. Let the invertible mapping $\hat{\mathbf{y}} : \mathcal{R} \to \mathcal{R}_*$, with

$$\hat{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{R}, \tag{2.1.1}$$

characterize a *deformation* of $\mathcal{B}$ from the reference configuration onto a configuration that occupies the region $\mathcal{R}_*$ in $I\!R^3$. Assume that the deformation $\hat{\mathbf{y}}$, or equivalently the *displacement* $\mathbf{u}$, is continuous and possesses piecewise continuous first and second gradients on $\mathcal{R}$. Let S be the set of points contained in $\mathcal{R}$ defined so that $\hat{\mathbf{y}}$ is differentiable on the set $\mathcal{R} \setminus S$. Introduce the *deformation gradient tensor* $\mathbf{F} : R \setminus S \to \mathcal{L}_+$ by

$$\mathbf{F}(\mathbf{x}) = \nabla \hat{\mathbf{y}}(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{R} \setminus S, \tag{2.1.2}$$

and assume that the associated *Jacobian determinant*, $J : \mathcal{R} \setminus S \to I\!R$, of $\hat{\mathbf{y}}$ is strictly positive on its domain of definition:

$$J(\mathbf{x}) = \det \mathbf{F}(\mathbf{x}) > 0 \quad \forall \mathbf{x} \in \mathcal{R} \setminus S. \tag{2.1.3}$$

The *left Cauchy-Green tensor* $\mathbf{G} : \mathcal{R} \setminus S \to \overset{+}{\mathcal{S}}$ corresponding to the deformation $\hat{\mathbf{y}}$ is given by

$$\mathbf{G}(\mathbf{x}) = \mathbf{F}(\mathbf{x})\mathbf{F}^T(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{R} \setminus S. \tag{2.1.4}$$

The *deformation invariants* associated with $\hat{\mathbf{y}}$ exist on $\mathcal{R} \setminus S$ and are supplied through the fundamental scalar invariants of $\mathbf{G}$:

$$I_1(\mathbf{G}) = \operatorname{tr} \mathbf{G}, \quad I_2(\mathbf{G}) = \tfrac{1}{2}\left((\operatorname{tr} \mathbf{G})^2 - \operatorname{tr}(\mathbf{G}^2)\right), \quad I_3(\mathbf{G}) = \det \mathbf{G}. \tag{2.1.5}$$

With the above kinematic antecedents in place introduce the *nominal mass density* $\rho : \mathcal{R} \to I\!R_+$, the *nominal body force per unit mass* $\mathbf{b} : \mathcal{R} \to I\!R^3$, and the *nominal stress tensor* $\mathbf{S} : \mathcal{R} \setminus S \to \mathcal{L}$, and suppose that ρ and $\mathbf{b}$ are continuous on $\mathcal{R}$, while $\mathbf{S}$ is continuous on $\mathcal{R} \setminus S$, piecewise continuous on $\mathcal{R}$, and also has a piecewise continuous gradient on $\mathcal{R}$. In the absence of constitutive assumptions relating the stress to the deformation gradient, the sets over which $\mathbf{S}$ and $\mathbf{F}$ suffer jumps need not be equivalent. The scope of this investigation is limited, however, to elastic materials wherein stress is continuously related to strain—hence, $\mathbf{S}$, like $\mathbf{F}$, is continuous on $\mathcal{R}\setminus S$. Let ρ_* be the mass density in the deformed configuration associated with $\hat{\mathbf{y}}$. Given a *regular* subregion $\mathcal{P}$ of $\mathcal{R}$, let $\mathbf{m} : \partial\mathcal{P} \to \mathcal{N}$ denote the unit outward normal to $\partial\mathcal{P}$. Then the global balance laws of mass, and—in the absence of inertia—force and moment equilibrium require that

$$\int_{\mathcal{P}} \rho \, dV = \int_{\hat{\mathbf{y}}(\mathcal{P})} \rho_* \, dV, \tag{2.1.6}$$

$$\int_{\partial\mathcal{P}} \mathbf{S}\mathbf{m} \, dA + \int_{\mathcal{P}} \rho\mathbf{b} \, dV = \mathbf{0}, \tag{2.1.7}$$

and

$$\int_{\partial\mathcal{P}} \hat{\mathbf{y}} \wedge \mathbf{S}\mathbf{m} \, dA + \int_{\mathcal{P}} \hat{\mathbf{y}} \wedge \rho\mathbf{b} \, dV = \mathbf{0}, \tag{2.1.8}$$

respectively, for every regular subregion $\mathcal{P}$ contained in $\mathcal{R}$.

Localization of the balance laws (2.1.6)-(2.1.8) at an arbitrary point contained in the interior of $\mathcal{R} \setminus S$ yields the following familiar field equations:

$$\begin{aligned} \rho = \rho_*(\hat{\mathbf{y}})J \quad &\text{on} \quad \mathcal{R} \setminus S, \\ \nabla \cdot \mathbf{S} + \rho\mathbf{b} = \mathbf{0} \quad &\text{on} \quad \mathcal{R} \setminus S, \\ \mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T \quad &\text{on} \quad \mathcal{R} \setminus S, \end{aligned} \tag{2.1.9}$$

Suppose, from now on, that the set S is a regular surface. Then, localization of (2.1.6)-(2.1.8) at an arbitrary point in S yields the following jump conditions:

$$\begin{aligned} [\![\rho_*(\hat{\mathbf{y}})J]\!] = 0 \quad &\text{on} \quad S, \\ [\![\mathbf{S}\mathbf{n}]\!] = \mathbf{0} \quad &\text{on} \quad S, \end{aligned} \tag{2.1.10}$$

where, given a generic field quantity $g : \mathcal{R} \setminus S \to I\!R$ which jumps across S, $[\![g]\!]$ is defined through

$$[\![g(\mathbf{x})]\!] = \lim_{h \searrow 0} \big(g(\mathbf{x} + h\mathbf{n}(\mathbf{x})) - g(\mathbf{x} - h\mathbf{n}(\mathbf{x}))\big) \quad \forall \mathbf{x} \in S, \tag{2.1.11}$$

with $\mathbf{n} : S \to \mathcal{N}$ a unit normal to S. Observe from the jump condition $(2.1.10)_1$ that the mass denity in the deformed state ρ_* is only defined on $\hat{\mathbf{y}}(\mathcal{R} \setminus S)$. Equations $(2.1.9)_1$ and $(2.1.10)_1$ are, evidently, completely decoupled from equations $(2.1.9)_{2,3}$ and $(2.1.10)_2$; that is, given a solution to, say, a boundary value problem involving the latter set of equations, ρ_* can be calculated *a posteriori*. For this reason equations $(2.1.9)_1$ and $(2.1.10)_1$ will be disregarded in the subsequent analysis. In addition to the jump conditions given in (2.1.10), the stipulated continuity of $\hat{\mathbf{y}}$ gives

$$[\![\mathbf{u}]\!] = \mathbf{0} \quad \text{on} \quad S. \tag{2.1.12}$$

2.2. Constitutive assumptions. Let $\mathcal{B}$ be composed of a hyperelastic material which is homogeneous, isotropic and incompressible. Since $\mathcal{B}$ is hyperelastic its mechanical response is governed by an elastic potential or *strain energy per unit reference volume*. The homogeneity of $\mathcal{B}$ implies that the elastic potential does not depend explicitly on position in the reference configuration. Furthermore, because $\mathcal{B}$ is isotropic the elastic potential can depend on the deformation gradient $\mathbf{F}$ only through the deformation invariants $I_k(\mathbf{G})$ defined in (2.1.7). The incompressibility of $\mathcal{B}$ requires that the deformation $\hat{\mathbf{y}}$ be *isochoric*, *i.e.*,

$$I_3(\mathbf{G}(\mathbf{x})) = J^2(\mathbf{x}) = 1 \quad \forall \mathbf{x} \in \mathcal{R} \setminus S. \tag{2.2.1}$$

An additional consequence of isotropy is, therefore, that the elastic potential can be expressed as a function solely of the first two deformation invariants. It can also be demonstrated via (2.1.5) that, when (2.1.1) holds, $I_\alpha(\mathbf{G}(\mathbf{x})) \geq 3$ for all $\mathbf{x}$ contained in $\mathcal{R} \setminus S$. Now, let $\check{W} : [3, \infty) \times [3, \infty) \to I\!R$ denote an elastic potential which characterizes $\mathcal{B}$ and assume that $\tilde{W}$ is continuously differentiable with piecewise continuous second derivatives on its domain of definition. The nominal stress response of $\mathcal{B}$ is then determined through $\tilde{W}$ up to an arbitrary pressure $p : \mathcal{R} \setminus S \to I\!R$ required to accomodate the kinematic constraint (2.2.1) imposed by the incompressibility of $\mathcal{B}$: *viz.*,

$$\mathbf{S} = 2\Big(\tilde{W}_{I_1}(I)\mathbf{F} + \tilde{W}_{I_2}(I)\big(I_1(\mathbf{G})\mathbf{1} - \mathbf{G}\big)\mathbf{G}\Big) - p\mathbf{F}^{-T} \quad \text{on} \quad \mathcal{R} \setminus S, \qquad (2.2.2)$$

where $I : \mathcal{R} \setminus S \to [3, \infty) \times [3, \infty)$ is given by

$$I(\mathbf{x}) = \big(I_1(\mathbf{G}(\mathbf{x})), I_2(\mathbf{G}(\mathbf{x}))\big) \quad \forall \mathbf{x} \in \mathcal{R} \setminus S.$$

Following GURTIN [16], let the class of *generalized neo-Hookean* materials refer to that subset of hyperelastic materials, first introduced by KNOWLES [19], which are homogeneous, isotropic and incompressible with elastic potential independent of the second deformation invariant $(2.1.5)_2$. Assume, henceforth, that $\mathcal{B}$ is composed of a generalized neo-Hookean material with elastic potential $W : [3, \infty) \to I\!R$, where W is continuously differentiable with piecewise continuous derivative on $[3, \infty)$. Then, by (2.2.2), the nominal stress response of $\mathcal{B}$ is determined by

$$\mathbf{S} = 2W'\big(I_1(\mathbf{G})\big)\mathbf{F} - p\mathbf{F}^{-T} \quad \text{on} \quad \mathcal{R} \setminus S. \qquad (2.2.3)$$

Suppose also that the elastic potential is *normalized* so that

$$W(3) = 0. \qquad (2.2.4)$$

Choose a rectangular Cartesian frame $X = \{0; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and consider the response of the material at hand to a simple shear deformation $\hat{\mathbf{y}}$ given by

$$\hat{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + \gamma(\mathbf{x} \cdot \mathbf{e}_2)\mathbf{e}_1 \quad \forall \mathbf{x} \in \mathcal{R}, \qquad (2.2.5)$$

where the constant γ—assumed non-negative without loss of generality—denotes the amount of shear. From (2.1.2), (2.2.3) and (2.2.5) the nominal shear stress corresponding to the deformation $\hat{\mathbf{y}}$ is, for each γ in $\bar{\mathbb{R}}_+$, found to be

$$\mathbf{e}_1 \cdot \mathbf{S}\mathbf{e}_2 = 2\gamma W'(3+\gamma^2) =: \tau(\gamma). \tag{2.2.6}$$

In [19–20] KNOWLES demonstrates that the components of nominal and Cauchy shear stress are, in the present setting, equal. The function $\tau : \bar{\mathbb{R}}_+ \to \mathbb{R}$ is, hence, referred to as the *shear stress response function* of the generalized neo-Hookean material, characterized by W, in simple shear. An immediate consequence of (2.2.4) and (2.2.6) is

$$W(I_1) = \int_0^{\sqrt{I_1-3}} \tau(\gamma)\, d\gamma \quad \forall I_1 \in [3,\infty), \tag{2.2.7}$$

so that the response of a generalized neo-Hookean material, in all three dimensional deformations, is, up to a hydrostatic pressure, completely characterized by specifying the shear stress response function τ. Define the *secant modulus in shear* $M : \bar{\mathbb{R}}_+ \to \mathbb{R}$ of a generalized neo-Hookean material with elastic potential W by

$$M(\gamma) = 2W'(3+\gamma^2) \quad \forall \gamma \in \bar{\mathbb{R}}_+, \tag{2.2.8}$$

and assume that, in compliance with the *Baker-Ericksen inequality*,

$$M(\gamma) > 0 \quad \forall \gamma \in \mathbb{R}_+. \tag{2.2.9}$$

Assume, also, that $M(0) > 0$ so that the infinitesimal shear modulus of the material at hand is positive. Note from (2.2.6) and (2.2.8) that the shear stress response function τ must also satisfy

$$\tau(0) = 0, \quad \tau'(0) = M(0). \tag{2.2.10}$$

Observe, also, that the stipulated smoothness of W guarantees that both τ and M are piecewise continuously differentiable on $\bar{\mathbb{R}}_+$.

It is worth remarking that, despite the significant restrictions which have been placed upon the class of materials which will be considered in this investigation, no presuppositions have been made regarding the sign of the derivative—where it exists—of the shear stress response function corresponding to the generalized neo-Hookean material through (2.2.6). In [20] KNOWLES shows that the monotonicity of the shear stress response function τ is related directly to the ellipticity of the generalized neo-Hookean material which it characterizes: if τ is not a monotone increasing function on its domain of definition then the associated material is non-elliptic. This investigation will make use of a particular subclass of non-elliptic generalized neo-Hookean materials, first suggested by ABEYARATNE [3]; this class of materials is characterized by the set of shear stress response functions τ which are continuous on $\bar{\mathbb{R}}_+$ and piecewise continuously differentiable on $\bar{\mathbb{R}}_+ \setminus \{\underset{*}{\gamma}, \overset{*}{\gamma}\}$, where $0 < \underset{*}{\gamma} < \overset{*}{\gamma}$, such that

$$\tau' > 0 \quad \text{on} \quad \bar{\mathbb{R}}_+ \setminus [\underset{*}{\gamma}, \overset{*}{\gamma}], \quad \tau' < 0 \quad \text{on} \quad (\underset{*}{\gamma}, \overset{*}{\gamma}). \tag{2.2.11}$$

The sets of shear strains lying in the intervals $[0, \underset{*}{\gamma})$ and $(\overset{*}{\gamma}, \infty)$ are referred to as the ***high*** and ***low strain phases*** of the generalized neo-Hookean material specified by the shear stress response function τ. Together the high and low strain phases of such a material comprise its ***elliptic phases.*** A generalized neo-Hookean material characterized by a shear stress response function of this type will be referred to herein as a ***three-phase*** material. See Figure 1 for a graph of a shear stress response function typical of those which specify three-phase materials. Within the class of three-phase materials special attention will be given those materials (proposed by ABEYARATNE in [2]) for which

$$\tau_p(\gamma) = \begin{cases} \mu_1 \gamma & \text{if} \quad \gamma \in [0, \underset{*}{\gamma}], \\ d(\gamma) & \text{if} \quad \gamma \in [\underset{*}{\gamma}, \overset{*}{\gamma}], \\ \mu_2 \gamma & \text{if} \quad \gamma \in [\overset{*}{\gamma}, \infty), \end{cases} \tag{2.2.12}$$

where the function $d : [\underset{*}{\gamma}, \overset{*}{\gamma}] \to I\!R$ is linear in its argument. Observe that, in accordance with $(2.2.11)_2$, d is required to decrease on $(\underset{*}{\gamma}, \overset{*}{\gamma})$. A further consequence of (2.2.11) is that μ_1 must be greater than μ_2 which must itself be positive. Figure 2 shows the graph of τ_p.

2.3. Dissipation, driving traction and the kinetic relation. For the purposes of this section it is necessary to consider a one parameter family of deformations $\hat{\mathbf{y}}(\cdot, t) : \mathcal{R} \to \mathcal{R}_t$ where t, which denotes time, increases from t_0 to t_1. It is assumed that $\hat{\mathbf{y}}(\mathbf{x}, \cdot)$ is continuous with piecewise continuous first and second derivatives for each fixed $\mathbf{x}$ in $\mathcal{R}$. Let S_t be a regular surface, with unit normal $\mathbf{n}(\cdot, t) : S_t \to \mathcal{N}$, contained in $\mathcal{R}_t$ for each value of t in $[t_0, t_1]$. The fields $\mathbf{u}(\cdot, t) : \mathcal{R} \to I\!R^3$, $\mathbf{F}(\cdot, t) : \mathcal{R} \setminus S_t \to \mathcal{L}_+$, $\mathbf{b}(\cdot, t) : \mathcal{R} \to I\!R^3$, and $\mathbf{S}(\cdot, t) : \mathcal{R} \setminus S_t \to \mathcal{L}$ are, at each t contained in $[t_0, t_1]$, the obvious counterparts of those introduced in Section 2.1. A one parameter family of deformations of this sort is referred to as a *quasistatic motion* if the above quantities and the nominal mass density satisfy the field equations

$$\begin{aligned} \nabla \cdot \mathbf{S}(\cdot, t) + \rho \mathbf{b}(\cdot, t) = \mathbf{0} \quad \text{on} \quad \mathcal{R} \setminus S_t \quad \forall t \in [t_0, t_1], \\ \mathbf{S}(\cdot, t)\mathbf{F}^T(\cdot, t) = \mathbf{F}(\cdot, t)\mathbf{S}^T(\cdot, t) \quad \text{on} \quad \mathcal{R} \setminus S_t \quad \forall t \in [t_0, t_1], \end{aligned} \tag{2.3.1}$$

the jump condition

$$[\![\mathbf{S}(\cdot, t)\mathbf{n}(\cdot, t)]\!] = \mathbf{0} \quad \text{on} \quad S_t \quad \forall t \in [t_0, t_1], \tag{2.3.3}$$

and the kinematic condition of displacement continuity

$$[\![\mathbf{u}(\cdot, t)]\!] = \mathbf{0} \quad \text{on} \quad S_t \quad \forall t \in [t_0, t_1]. \tag{2.3.4}$$

KNOWLES [18] has shown that, in a quasistatic motion, the presence of a moving surface of discontinuity S_t of the type considered here has an effect on the balance of mechanical energy. Let $\mathcal{P}$ be a regular subregion contained in $\mathcal{R}$.

In [18] it is demonstrated that the difference in the rate of work of the mechanical forces external to $\mathcal{P}$ and the rate at which energy is stored in $\mathcal{P}$ is given by

$$\delta_s(t;\mathcal{P}) = \int\limits_{S_t \cap \mathcal{P}} f(\mathbf{x},t)V_n(\mathbf{x},t)\,dA \quad \forall t \in [t_0, t_1], \tag{2.3.5}$$

where, for each t in $[t_0, t_1]$, $f(\cdot,t) : S_t \to I\!R$ is the scalar *driving traction* and $V_n(\cdot,t) : S_t \to I\!R$ is the *normal velocity* of the interface (in the reference configuration). The function $\delta_s(\cdot;\mathcal{P}) : [t_0,t_1] \to I\!R$ is referred to as the *rate of dissipation* of mechanical energy associated with the region $\mathcal{P}$. It has been shown by YATOMI & NISHIMURA [26] as well as ABEYARATNE & KNOWLES [7] that the form of the driving traction for a hyperelastic material is, in the quasistatic setting, supplied by

$$f(\cdot,t) = [\![W(\mathbf{F}(\cdot,t))]\!] - \overset{\pm}{\mathbf{S}}(\cdot,t)\cdot[\![\mathbf{F}(\cdot,t)]\!] \quad \text{on} \quad S_t \quad \forall t \in [t_0,t_1], \tag{2.3.6}$$

where $\overset{+}{\mathbf{S}}(\cdot,t)$ (*resp.*, $\overset{-}{\mathbf{S}}(\cdot,t)$) is the limiting value of the field $\mathbf{S}(\cdot,t)$ on the side of the interface into which the unit normal $\mathbf{n}(\cdot,t)$ is (*resp.*, is not) directed at t in $[t_0,t_1]$.

When treated from a *thermomechanical* perspective, the dissipation rate can be shown to be identical to the product of the temperature and the rate of entropy production—provided that the temperature is spatially uniform and independent of time.[2] The *Clausius-Duhem inequality* then requires that the dissipation rate associated with a quasistatic motion of the kind envisioned here be non-negative, that is

$$\delta_s(t;\mathcal{P}) \geq 0 \quad \forall t \in [t_0,t_1], \tag{2.3.7}$$

for every regular subregion $\mathcal{P}$ contained in $\mathcal{R}$. A localization of (2.3.5) at an arbitrary point on the interface therefore yields the inequality

$$f(\cdot,t)V_n(\cdot,t) \geq 0 \quad \text{on} \quad S_t \quad \forall t \in [t_0,t_1] \tag{2.3.8}$$

[2] For a detailed discussion of the these issues see ABEYARATNE & KNOWLES [7].

as a condition imposed for the ***admissibility*** of the quasistatic motion.

In the context of a motion which involves such an interface it is necessary (see [1] and [3–8]) to supplement, in some fashion, the constitutive information which relates the stress and strain fields. An approach to this pioneered by ABEYARATNE & KNOWLES [7] entails the provision of a ***kinetic relation*** which gives the normal velocity of the interface in terms of the driving traction which acts thereon or ***vice versa***. In the former case one specifies a function $\tilde{V} : \mathbb{R} \to \mathbb{R}$ such that

$$V_n = \tilde{V}(f) \quad \forall f \in \mathbb{R}. \tag{2.3.9}$$

Here $\tilde{V}$ is referred to as the ***kinetic response function***. If the function $\tilde{V}$ is such that $\tilde{V}(f)f \geq 0$ on $\mathbb{R}$ then (2.3.8) is automatically satisfied and the kinetic response function is itself referred to as ***admissible***. If an admissible kinetic response function is continuous on $\mathbb{R}$, then it must satisfy $\tilde{V}(0) = 0$. If, in addition to being admissible, $\tilde{V}$ is continuously differentiable on $\mathbb{R}$, then $\tilde{V}'(0) \geq 0$. Otherwise admissibility implies nothing with regard to the sign of the derivative of a smooth kinetic response function $\tilde{V}$. All kinetic response functions considered herein are assumed to be admissible.

In the work of ABEYARATNE [3], BALL & JAMES [10], GURTIN [16], GURTIN & TEMAM [15], and SILLING [25] the necessary additional constitutive information is provided by setting the driving traction equal to zero on S_t for all t in $[t_0, t_1]$. This amounts to prescribing a particular rate independent kinetic relation whereby energy is conserved; it is, furthermore, a necessary consequence of requiring that a suitable ***energy functional*** be minimized at each t in $[t_0, t_1]$ (see ABEYARATNE [1]).

2.4. Antiplane shear of a generalized neo-Hookean material. Suppose, from now on, that $\mathcal{R}$ is a cylindrical region and choose a rectangular Cartesian frame $X = \{0; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ so that the unit base vector $\mathbf{e}_3$ is parallel to the generatrix of $\mathcal{R}$. The deformation $\hat{\mathbf{y}}$ defined through (2.1.1) consists of an antiplane shear if it is of the form

$$\hat{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + u(x_1, x_2)\mathbf{e}_3 \quad \forall \mathbf{x} \in \mathcal{R}, \tag{2.4.1}$$

That is, the displacement field intrinsic to an antiplane shear deformation has only one nonzero component in the $\mathbf{e}_3$ direction which is independent of the x_3–coordinate. In (2.4.1) $x_\alpha = \mathbf{x} \cdot \mathbf{e}_\alpha$ for each $\mathbf{x}$ contained in $\mathcal{R}$. The function u will be referred to as the *out-of-plane* displacement field. Inspection of (2.4.1) reveals that any discontinuities in the gradient of $\hat{\mathbf{y}}$ must be due to discontinuities in the out-of-plane displacement field and, hence, occur across surfaces which do not vary with the x_3–coordinate.

KNOWLES [19] has demonstrated that, although not every hyperelastic, isotropic and incompressible material can sustain antiplane shear deformations, all generalized neo-Hookean materials are capable of doing so. It has been shown (KNOWLES [19–20]) that for such materials the local balance equations $(2.1.9)_{2,3}$ reduce, in the absense of body forces and under the asssumption that the nominal stress tensor is independent of the x_3-coordinate, to the scalar equation

$$\big(M(\gamma)u_{,\alpha}\big)_{,\alpha} = 0 \quad \text{on} \quad \mathcal{D} \setminus C. \tag{2.4.2}$$

Here M is the secant modulus in shear as defined in (2.2.8), $\gamma : \mathcal{D} \setminus C \to I\!R$ is the *shear strain field* given by

$$\gamma(x_1, x_2) = \sqrt{u_{,\alpha}(x_1, x_2)u_{,\alpha}(x_1, x_2)} \quad \forall (x_1, x_2) \in \mathcal{D} \setminus C, \tag{2.4.3}$$

$\mathcal{D}$ is a plane region with shape determined by a generic cross section of $\mathcal{R}$, and C is a curve contained in $\mathcal{D}$ and determined similarly by a cross section of the surface across which the deformation gradient jumps. Furthermore, the jump condition $(2.1.10)_2$ reduces, for a generalized neo-Hookean material subjected to antiplane shear, to

$$[\![M(\gamma)u_{,\alpha}n_\alpha]\!] = 0 \quad \text{on} \quad C, \tag{2.4.4}$$

where $\mathbf{n} : C \to \mathcal{N}$ is a unit normal to C, while (2.1.15) becomes

$$[\![u]\!] = 0 \quad \text{on} \quad C. \tag{2.4.5}$$

It is also readily shown that the driving traction f, introduced in Section 2.3, for a generalized neo-Hookean material subjected to an antiplane shear deformation involving a discontinuity in the gradient of displacement across a curve C is given by

$$f = [\![W(3+\gamma^2)]\!] - M(\overset{\pm}{\gamma})\overset{\pm}{u}_{,\alpha}\,[\![u_{,\alpha}]\!] \quad \text{on} \quad C. \tag{2.4.6}$$

In (2.4.6) $\overset{+}{u}_{,\alpha}$ and $\overset{-}{u}_{,\alpha}$ refer to the limiting values of the gradient of the out-of-plane displacement field on the side of the curve C into which and out of which the unit normal $\mathbf{n}$ points, respectively. Evidently, $\overset{+}{\gamma}$ and $\overset{-}{\gamma}$ are given in terms of $\overset{+}{u}_{,\alpha}$ and $\overset{-}{u}_{,\alpha}$ by (2.4.3).

3. STUDY OF A CONSTRAINED BOUNDARY VALUE PROBLEM IN THE ANTIPLANE SHEAR OF A THREE-PHASE MATERIAL

This chapter focuses, in the context of antiplane shear, on the construction of two-phase equilibria of a body composed of the particular non-elliptic generalized neo-Hookean material with shear stress response function τ_p defined via (2.2.12). These equilibria will involve non-planar interfaces which segregate material in different elliptic phases. The interfaces will be described by surfaces $\mathcal{Q}_s$ of the form

$$\mathcal{Q}_s = \{\mathbf{x} \in \mathbb{R}^3 \,|\, x_1 = s(x_2), x_2 \in \mathbb{R}, x_3 \in \mathbb{R}\}$$

where s is twice continuously differentiable on $\mathbb{R}$, $s^{(n)}$ is in $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ for $n = 0, 1, 2$, and

$$\lim_{x_2 \to \pm\infty} s(x_2) = 0.$$

In Section 3.1 a boundary value problem for the out-of-plane displacement field associated with two-phase antiplane shear deformations of a three-phase material is formulated and specialized to the case of the material with shear stress response τ_p. This boundary value problem is supplemented by a set of constraints which require that the non-elliptic phase of the relevant material is absent and, moreover, that the elliptic phases are segregated. In Section 3.2 the boundary value problem is converted into an integral equation for the jump in the normal derivative of the out-of-plane displacement field across $\mathcal{Q}_s$. In Section 3.3 it is shown that there exists a unique solution to this integral equation for every $\mathcal{Q}_s$ defined by a function s which, in addition to the restrictions delineated above, satisfies

$$\Big(\int_{-\infty}^{+\infty} |s'(x_2)||s''(x_2)|\, dx_2\Big)^{\frac{1}{2}} < \sqrt{\tfrac{3\pi}{2}}\,\frac{\mu_1 + \mu_2}{\mu_1 - \mu_2},$$

where μ_1 and μ_2 are the moduli associated with the elliptic phases of the material defined by τ_p. It is then demonstrated that for each such s there exists a unique (up to an arbitrary additive constant) solution to the boundary value problem stated in Section 3.1. Since, however, the constraints mentioned above are not necessarily satisfied by any of these solutions, the deformation associated with a given solution does not necessarily constitute a two-phase equilibrium state of the type sought after here. In Section 3.4 it is shown that, provided a certain *functional* of s is sufficiently small—in a sense to be made precise—then there exists a unique solution to the constrained boundary value problem and, hence, a two-phase equilibrium state of the type sought after here. The concluding section of this chapter is concerned with the construction of a class of two-phase states which involve non-planar interfaces separating material in distinct elliptic phases.

3.1. Formulation and reduction of the boundary value problem and phase segregation requirements. Suppose that $\mathcal{B}$ is composed of a three-phase material and that the cylinder $\mathcal{R}$ degenerates to occupy all of $\mathbb{R}^3$. Let the rectangular Cartesian frame X be as in Section 2.4. Consider the effect of subjecting $\mathcal{R}$ to a particular antiplane loading whereby, independent of the x_2–coordinate, the shear strain approaches uniform values of γ_l as x_1 tends to $-\infty$ and γ_r as x_1 tends to $+\infty$. Assume that γ_l is greater than $\overset{*}{\gamma}$ and that γ_r lies strictly between 0 and $\underset{*}{\gamma}$; the prescribed remote shear strains associated with the loading are, thus, in the high and low strain phases of the material at hand as x_1 approaches $-\infty$ and $+\infty$, respectively. If γ_l and γ_r are chosen so that the corresponding remote shear stresses $\tau(\gamma_l)$ and $\tau(\gamma_r)$ are equal then—for every three-phase material—there exists, modulo an arbitrary additive constant, a unique one parameter family of pairwise homogeneous out-of-plane displacement fields $u_a : \mathbb{R} \to \mathbb{R}$ which satisfy the equilibrium equation in (2.4.2) on $\mathbb{R}^2 \setminus C_a$, with the straight line C_a given by $\{x_\alpha \mathbf{e}_\alpha \in \mathbb{R}^2 | x_1 = a, x_2 \in \mathbb{R}\}$, the jump conditions in (2.4.4) and (2.4.5) on C_a and, of course, the decay requirements associated with the prescribed conditions at $x_1 = \pm\infty$. The function u_a is given by

$$u_a(x_1) = \begin{cases} \gamma_l(x_1 - a) & \text{if} \quad x_1 < a, \\ \gamma_r(x_1 - a) & \text{if} \quad x_1 > a. \end{cases} \tag{3.1.1}$$

Here a determines the point of intersection of the plane surface Q_a, given by $C_a \times \mathbb{R}$, with the x_1–axis. Note that, for each fixed a, the pairwise homogeneous deformation associated with (3.1.1) through (2.4.1) involves exclusively the elliptic phases of the material under consideration and that these are segregated by Q_a; the deformation associated with (3.1.1) will be referred to as a *globally elliptic pairwise homogeneous equilibrium state.* The interface Q_a associated with such a state will, in turn, be referred to as a *phase boundary.* Observe that the qualitative character of the equilibrium state associated with u_a is clearly unaffected by the value of a.

Envision a generalization of the globally elliptic pairwise homogeneous equilibrium state wherein the kinematics remain those of antiplane shear and the loading conditions are as described at the outset of this section but the planar phase boundary is replaced by a non-planar interface Q_s with cross section C_s where, for simplicity,

$$Q_s = C_s \times \mathbb{R} \tag{3.1.2}$$

with

$$C_s = \{x_\alpha \mathbf{e}_\alpha \in \mathbb{R}^2 \,|\, x_1 = s(x_2), x_2 \in \mathbb{R}\}. \tag{3.1.3}$$

Assume that the state is equilibrated in the sense that the balance equation in (2.4.2) holds on $\mathbb{R}^2 \setminus C_s$ while the jump conditions in (2.4.4) and (2.4.5) are satisfied on C_s. Clearly, if such a state exists, the deformation field intrinsic to it must be inhomogeneous on either side of the interface Q_s. Observe that even if a three-phase material is capable of sustaining a deformation of this kind the shear strain field may not, in general, be distributed so that only the elliptic phases of the material are present; if, however, this is the case and, furthermore, the high and low strain phases of the relevant material are segregated by the interface Q_s then the deformation will be said to constitute a *globally elliptic inhomogeneous two-phase equilibrium state* with phase boundary Q_s.

Consider, now, the geometry of the curve C_s which determines the phase boundary Q_s essential to a globally elliptic inhomogeneous two-phase equilibrium state. Since the shear strain field is constant as x_1 approaches $\pm\infty$, it is clear that s must be bounded on $\mathbb{R}$ in order for $\mathcal{C}_s$ to qualify as a cross section of the phase boundary $\mathcal{Q}_s$. The kinematics and boundary conditions place no further restrictions on the geometry of $\mathcal{C}_s$.

Observe that if, in addition to being bounded and continuous on $\mathbb{R}$, s satisfies one or both of

$$\lim_{x_2 \to -\infty} s(x_2) = \overset{-}{c}, \qquad \lim_{x_2 \to +\infty} s(x_2) = \overset{+}{c}, \tag{3.1.4}$$

where $\overset{-}{c}$ and $\overset{+}{c}$ are real constants, then the loading must be restricted so that the far field shear stresses $\tau(\gamma_l)$ and $\tau(\gamma_r)$ are equal. To see this suppose that $(3.1.4)_1$ holds. Then, as x_2 approaches $-\infty$ the phase boundary becomes planar and the local character of the deformation begins to resemble a pairwise homogeneous state. Since the far field shear strains γ_l and γ_r are constant the local shear strains must match these appropriately on either side of $\mathcal{C}_s$ as x_1 approaches $-\infty$. Hence, the local shear stresses must match their far field counterparts $\tau(\gamma_l)$ and $\tau(\gamma_r)$ and, by the jump condition in (2.4.4) which holds on $\mathcal{C}_s$, $\tau(\gamma_l) = \tau(\gamma_r)$. A completely analogous argument can be constructed if $(3.1.4)_2$ holds instead of $(3.1.4)_1$. Certainly, if both of (3.1.4) hold, the result is still true. Note, however, that if neither of (3.1.4) hold, and, hence, the curve $\mathcal{C}_s$ is merely bounded, there is no reason to rule out—*a priori*—loading conditions wherein the far field stresses are unequal.

Assume, henceforth, that (3.1.4) holds with $\overset{-}{c}$ and $\overset{+}{c}$ equal to, say, c. Recalling the role of a in (3.1.1), there is certainly no additional loss in generality incurred by taking $c = 0$. In this case (3.1.4) becomes

$$\lim_{x_2 \to \pm\infty} s(x_2) = 0. \tag{3.1.5}$$

Let $\mathcal{U}$ be the set of functions defined by

$$\mathcal{U} = \{\psi : \mathbb{R} \to \mathbb{R} \,|\, \psi \in C(\mathbb{R}), \lim_{x_2 \to \pm\infty} \psi(x_2) = 0, \psi \not\equiv 0 \text{ on } \mathbb{R}\}.$$

Assume, henceforth, that s is an element of the set

$$\mathcal{A} = \mathcal{U} \cap \mathcal{V} \cap \mathcal{W}, \tag{3.1.6}$$

where $\mathcal{V}$ and $\mathcal{W}$ are defined by

$$\mathcal{V} = \{\psi : I\!R \to I\!R \,|\, \psi \in C^2(I\!R), \psi^{(n)} \in L^2(I\!R), n = 0, 1, 2\}, \tag{3.1.7}$$

and

$$\mathcal{W} = \{\psi : I\!R \to I\!R \,|\, \psi \in C^2(I\!R), \psi^{(n)} \in L^\infty(I\!R), n = 0, 1, 2\}, \tag{3.1.8}$$

respectively.

Given an element s of $\mathcal{A}$ which describes an interface $\mathcal{Q}_s$ it is convenient to define plane sets $\mathcal{D}_s^l$ and $\mathcal{D}_s^r$ by

$$\mathcal{D}_s^l = \{x_\alpha \mathbf{e}_\alpha \in I\!R^2 \,|\, x_1 \leq s(x_2), x_2 \in I\!R\}, \quad \mathcal{D}_s^r = I\!R^2 \setminus \mathring{\mathcal{D}}_s^l. \tag{3.1.9}$$

Clearly, the union and intersection of $\mathcal{D}_s^l$ and $\mathcal{D}_s^r$ form generic cross-sections of the cylinder $\mathcal{R}$ and the phase boundary $\mathcal{Q}_s$, respectively. Note, also, that if s is an element of $\mathcal{A}$ then, by (3.1.3) and its assumed smoothness, a unit normal to $\mathcal{Q}_s$ exists everywhere on $\mathcal{Q}_s$ and depends only on the x_2–coordinate. Let $\mathbf{n} : I\!R \to \mathcal{N}$ designate the unit normal to $\mathcal{Q}_s$ which points into the region of low strain—$\mathring{\mathcal{D}}_s^r \times I\!R$. Then the representation for $\mathbf{n}$ is computed easily from the definitions of $\mathcal{Q}_s$ and $\mathcal{C}_s$ and is given by

$$\mathbf{n}(x_2) = \frac{\mathbf{e}_1 - s'(x_2)\mathbf{e}_2}{\sqrt{1 + s'(x_2)^2}} \quad \forall x_2 \in I\!R. \tag{3.1.10}$$

Now, if a three-phase material is capable of sustaining a globally elliptic inhomogeneous two-phase equilibrium state of antiplane shear with phase boundary $\mathcal{Q}_s$ then the out-of-plane displacement field u associated with the deformation through (2.4.1) must, by virtue of (2.4.2), (2.4.4) and (2.4.5), satisfy the following field equation and jump conditions:

$$
\begin{aligned}
(M(\gamma)u_{,\alpha})_{,\alpha} &= 0 \quad \text{on} \quad I\!R^2 \setminus \mathcal{C}_s, \\
[\![M(\gamma)u_{,\alpha} n_\alpha]\!] &= 0 \quad \text{on} \quad \mathcal{C}_s, \\
[\![u]\!] &= 0 \quad \text{on} \quad \mathcal{C}_s,
\end{aligned}
\tag{3.1.11}
$$

with $\mathbf{n}$ as indicated in (3.1.10); in order to comply with the prescribed loading it suffices to require that the gradient of u satisfies the following asymptotic decay conditions:

$$
u_{,\alpha}(x_1,\cdot)\mathbf{e}_\alpha = \begin{cases} \gamma_l \mathbf{e}_1 + \mathbf{o}(1) & \text{as} \quad x_1 \to -\infty, \\ \gamma_r \mathbf{e}_1 + \mathbf{o}(1) & \text{as} \quad x_1 \to +\infty, \end{cases} \quad \text{on} \quad I\!R; \tag{3.1.12}
$$

moreover, in order to assure that only the elliptic phases of the material at hand are present and are segregated by $\mathcal{Q}_s$, the shear strain field γ, given in terms of the gradient of u by (2.4.3), must conform to the following inequalities:

$$
\gamma \in (\overset{*}{\gamma}, \infty) \quad \text{on} \quad \mathring{\mathcal{D}}_s^l, \quad \gamma \in [0, \underset{*}{\gamma}) \quad \text{on} \quad \mathring{\mathcal{D}}_s^r, \tag{3.1.13}
$$

where $\mathcal{D}_s^l$ and $\mathcal{D}_s^r$ are given by (3.1.9). These inequalities will be referred to as the *phase segregation requirement.*

Given a three-phase material, (3.1.11)-(3.1.12) comprise, for each fixed s contained in $\mathcal{A}$, a boundary value problem in the out-of-plane displacement field u, while (3.1.13) acts as a system of constraints thereon. Together (3.1.11)-(3.1.13) will be referred to as the constrained boundary value problem in u for the three-phase material with secant modulus in shear M. Given a particular three-phase material the constrained boundary value problem need not have a solution for any function s in $\mathcal{A}$. The study of (3.1.11)-(3.1.13) for a specific material may, however, serve as a means to determine a subset of $\mathcal{A}$ for which the constrained boundary value problem is soluble.

Before proceeding note that the jump conditions $(3.1.11)_{2,3}$ holding across $\mathcal{C}_s$ can be recast to read

$$\begin{gathered} M\left(\gamma\left(s(\cdot)+,\cdot\right)\right)\frac{\partial u}{\partial n}\left(s(\cdot)+,\cdot\right) = M\left(\gamma\left(s(\cdot)-,\cdot\right)\right)\frac{\partial u}{\partial n}\left(s(\cdot)-,\cdot\right) \quad \text{on} \quad I\!\!R, \\ u\left(s(\cdot)+,\cdot\right) = u\left(s(\cdot)-,\cdot\right) \quad \text{on} \quad I\!\!R, \end{gathered} \tag{3.1.14}$$

where the $+$ and $-$ symbols indicate the limiting values of the appropriate quantities on the high and low strain sides of the interface, respectively.

For simplicity attention will, for the remainder of this work, be restricted to the constrained boundary value problem for the material characterized by the shear stress response function τ_p defined in (2.2.12). In this case the form of the shear stress response function is such that the secant modulus in shear M is constant in both the high and low strain elliptic phases; hence, $(3.1.11)_1$ and $(3.1.14)_1$ reduce to

$$\begin{gathered} u_{,\alpha\alpha} = 0 \quad \text{on} \quad I\!\!R^2 \setminus \mathcal{C}_s, \\ \mu_1 \frac{\partial u}{\partial n}\left(s(\cdot)+,\cdot\right) = \mu_2 \frac{\partial u}{\partial n}\left(s(\cdot)-,\cdot\right) \quad \text{on} \quad I\!\!R. \end{gathered} \tag{3.1.15}$$

The analytical difficulties of the special constrained boundary value problem posed by (3.1.15), $(3.1.14)_2$, (3.1.12) and (3.1.13) are certainly less daunting than those encountered in the analogous problem for a more general three-phase material. For each fixed s in $\mathcal{A}$ the only non-linearity which encumbers the problem associated with τ_p is that imposed by the strain constraints (3.1.13). In the present absence of results pertaining to the existence of globally elliptic two-phase equilibria in arbitrary three-phase materials, any results which can be obtained for this particular material constitute progress toward a qualitative understanding of the more general issue.

As a first step in analyzing the constrained boundary value problem comprised by (3.1.15), $(3.1.14)_2$, (3.1.12) and (3.1.13) it is convenient to introduce a *reduced* out-of-plane displacement field $v : I\!\!R^2 \setminus \mathcal{C}_s \to I\!\!R$ specified via

$$v(x_1,x_2) = u(x_1,x_2) - u_0(x_1,x_2) \quad \forall (x_1,x_2) \in I\!\!R^2 \setminus \mathcal{C}_s, \tag{3.1.16}$$

where $u_0 : \mathbb{R}^2 \setminus \mathcal{C}_s \to \mathbb{R}$ is furnished by

$$u_0(x_1, x_2) = H\left(s(x_2) - x_1\right)\gamma_l x_1 + H\left(x_1 - s(x_2)\right)\gamma_r x_1 \quad \forall (x_1, x_2) \in \mathbb{R}^2 \setminus \mathcal{C}_s,$$

and $H : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the *Heavyside function*:

$$H(x_1) = \begin{cases} 0 & \text{if} \quad x_1 < 0, \\ 1 & \text{if} \quad x_1 > 0. \end{cases}$$

Solving for u in (3.1.16) and inserting the result in (3.1.15), $(3.1.14)_2$ and (3.1.12) shows, with the aid of the definitions of u_0 and H, that the reduced out-of-plane displacement field v must satisfy the following boundary value problem:

$$\begin{aligned} v_{,\alpha\alpha} &= 0 \quad \text{on} \quad \mathbb{R}^2 \setminus \mathcal{C}_s, \\ \mu_1 \frac{\partial v}{\partial n}(s(\cdot)+, \cdot) &= \mu_2 \frac{\partial v}{\partial n}(s(\cdot)-, \cdot) \quad \text{on} \quad \mathbb{R}, \\ v(s(\cdot)+, \cdot) - v(s(\cdot)-, \cdot) &= (\gamma_l - \gamma_r)s \quad \text{on} \quad \mathbb{R}, \\ v_{,\alpha}(x_1, \cdot)\mathbf{e}_\alpha &= \mathbf{o}(1) \quad \text{as} \quad x_1 \to \pm\infty \quad \text{on} \quad \mathbb{R}. \end{aligned} \tag{3.1.17}$$

Note that in deriving $(3.1.17)_2$ use has been made of the equality of remote shear stresses—which, as shown at the beginning of this section, is a necessary consequence of (3.1.5). The phase segregation requirement (3.1.13) can be written—after appropriate substitution for u—in terms of the components of the gradient of v as follows:

$$\begin{aligned} \overset{*}{\gamma}{}^2 &< v_{,\alpha} v_{,\alpha} + \gamma_l(2v_{,1} + \gamma_l) \quad \text{on} \quad \mathring{\mathcal{D}}_s^l, \\ 0 \le v_{,\alpha} v_{,\alpha} &+ \gamma_r(2v_{,1} + \gamma_r) < \underset{*}{\gamma}^2 \quad \text{on} \quad \mathring{\mathcal{D}}_s^r. \end{aligned} \tag{3.1.18}$$

The boundary value problem in (3.1.17) will be referred to as the ***reduced boundary value problem*** with the implicit understanding that it is in the reduced out-of-plane displacement field v and for the special three-phase material with shear stress response function τ_p. The system of inequalities in (3.1.18) will be labelled the *reduced phase segregation requirement.* It is clear from the simple relation between the primitive out-of-plane displacement field u and its reduced counterpart v that any solution to the reduced problem yields a solution to the original problem. The next section will focus on obtaining a representation for the solution to the reduced boundary value problem with the reduced phase segregation requirement held in abeyance. This representation will lead to an integral equation which, for each fixed s in $\mathcal{A}$, may be analyzed in place of the associated reduced boundary value problem.

3.2. Reformulation of the reduced boundary value problem as an integral equation. Let s be an arbitrary element of $\mathcal{A}$. Since, by $(3.1.17)_1$, the reduced out-of-plane displacement field v is *harmonic* on $\mathbb{R}^2 \setminus \mathcal{C}_s$ the jump conditions $(3.1.17)_2$ and $(3.1.17)_3$ suggest that v can be represented, modulo an arbitrary additive constant, as the sum of a *single-* and a *double-layer potential* along the curve $\mathcal{C}_s$.[3] The densities of the appropriate single- and double-layer potentials are given, respectively, in terms of the jumps in the normal derivative of v and of v itself across the curve $\mathcal{C}_s$. From the jump condition $(3.1.17)_3$ it is clear that the density of the double-layer potential must be given by $(\gamma_l - \gamma_r)s$ on $\mathbb{R}$. Since, by the definition of the shear stress response function τ_p, the moduli μ_1 and μ_2 which appear in $(3.1.17)_2$ are required to be unequal, this jump condition does not yield direct information regarding the form of the density of the single-layer potential. It is, therefore, necessary to designate the jump in the normal derivative of v across $\mathcal{C}_s$ in terms of an unknown function—say $(\gamma_l - \gamma_r)\phi$, where it is assumed, until demonstrated otherwise, that $\phi : \mathbb{R} \to \mathbb{R}$ does not vanish identically on $\mathbb{R}$. Hence, the proposed representation for the reduced out-of-plane displacement field v takes the form

$$v(x_1, x_2) = \frac{\gamma_l - \gamma_r}{2\pi} \left(S_\phi(x_1, x_2) + D_s(x_1, x_2) \right) \quad \forall (x_1, x_2) \in \mathbb{R}^2 \setminus \mathcal{C}_s, \qquad (3.2.1)$$

where the functions $S_\phi : \mathbb{R}^2 \to \mathbb{R}$ and $D_s : \mathbb{R}^2 \setminus \mathcal{C}_s \to \mathbb{R}$ issue, respectively,

[3] For an overview of the relevant potential theory see COURANT & HILBERT [11].

from the single- and double-layer potentials on $\mathcal{C}_s$ with densities ϕ and s and are given by

$$S_\phi(x_1,x_2) = \int_{-\infty}^{+\infty} G_1^s(x_1,x_2,\xi)\phi(\xi)\sqrt{1+s'(\xi)^2}\,d\xi \quad \forall (x_1,x_2) \in \mathbb{R}^2, \tag{3.2.2}$$

and

$$D_s(x_1,x_2) = \int_{-\infty}^{+\infty} G_2^s(x_1,x_2,\xi)s(\xi)\,d\xi \quad \forall (x_1,x_2) \in \mathbb{R}^2 \setminus \mathcal{C}_s. \tag{3.2.3}$$

The kernels $G_1^s : (\mathbb{R}^2 \setminus \mathcal{C}_s) \times \mathbb{R} \to \mathbb{R}$ and $G_2^s : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ which appear in (3.2.2) and (3.2.3) are given, for each ξ contained in $\mathbb{R}$, by

$$G_1^s(x_1,x_2,\xi) = \ln\sqrt{(x_1 - s(\xi))^2 + (x_2-\xi)^2} \quad \forall (x_1,x_2) \in \mathbb{R}^2 \setminus \mathcal{C}_s, \tag{3.2.4}$$

and

$$G_2^s(x_1,x_2,\xi) = \frac{(x_1 - s(\xi)) - (x_2 - \xi)s'(\xi)}{(x_1 - s(\xi))^2 + (x_2-\xi)^2} \quad \forall (x_1,x_2) \in \mathbb{R}^2. \tag{3.2.5}$$

Consider, now, the issue of verifying the status of the representation (3.2.1) as a solution to the boundary value problem (3.1.17). Since the single- and double-layer potentials are harmonic, by construction, on $\mathbb{R}^2 \setminus \mathcal{C}_s$, it is evident that the function v given by (3.2.1)–(3.2.5) satisfies $(3.1.17)_1$. A series of direct calculations too long to display here show that $v_{,1}(x_1,\cdot)$ and $v_{,2}(x_1,\cdot)$ both behave asymptotically like $O(1/x_1)$ as x_1 approaches $\pm\infty$ on $\mathbb{R}$ so that the representation (3.2.1)–(3.2.5) complies with $(3.1.17)_4$ and, hence, the loading conditions. Since the single-layer term (3.2.2) is continuous on $\mathbb{R}^2$ and the double-layer term (3.2.3) has been constructed so that is possesses a jump of $2\pi s$ across the curve $\mathcal{C}_s$ it is also clear that (3.2.1)–(3.2.5) furnishes a representation of v which satisfies the jump condition in $(3.1.17)_3$. The only remaining requirement which must be satisfied by (3.2.1)–(3.2.5) in order for it to provide a solution to the reduced boundary value problem is the jump condition $(3.1.17)_2$ involving the normal derivative of v. A straightforward but tedious calculation using standard results from potential theory delivers the limits of the normal derivative of v on either side of $\mathcal{C}_s$ in the form

$$\frac{\partial v}{\partial n}(s(x_2)\pm, x_2) = -\frac{\gamma_l - \gamma_r}{2\pi\sqrt{1+s'(x_2)^2}} \fint_{-\infty}^{+\infty} I_s(x_2,\xi)s'(\xi)\,d\xi$$
$$+\frac{\gamma_l - \gamma_r}{2\pi\sqrt{1+s'(x_2)^2}} \int_{-\infty}^{+\infty} K_s(x_2,\xi)\phi(\xi)\sqrt{1+s'(\xi)^2}\,d\xi$$
$$\pm\frac{\gamma_l - \gamma_r}{2}\phi(x_2) \quad \forall x_2 \in \mathbb{R}, \tag{3.2.6}$$

where, for each fixed x_2 in $\mathbb{R}$, $I_s(x_2,\cdot) : \mathbb{R}\setminus\{x_2\} \to \mathbb{R}$ and $K_s(x_2,\cdot) : \mathbb{R} \to \mathbb{R}$ are given, repectively, by

$$I_s(x_2,\xi) = \frac{(s(x_2)-s(\xi))s'(x_2) + (x_2-\xi)}{(s(x_2)-s(\xi))^2 + (x_2-\xi)^2} \quad \forall \xi \in \mathbb{R}\setminus\{x_2\}, \tag{3.2.7}$$

and

$$K_s(x_2,\xi) = \frac{(s(x_2)-s(\xi)) - (x_2-\xi)s'(x_2)}{(s(x_2)-s(\xi))^2 + (x_2-\xi)^2} \quad \forall \xi \in \mathbb{R}. \tag{3.2.8}$$

Observe from (3.2.8) that as ξ approaches x_2, $I_s(x_2,\xi)$ is singular for each x_2 in $\mathbb{R}$ in that

$$I_s(x_2,\xi) \sim \frac{1}{x_2-\xi} \quad \text{as} \quad \xi \to x_2 \quad \forall x_2 \in \mathbb{R}.$$

Hence, the integral involving I_s in (3.2.6) must, as indicated, be taken in the sense of the *Cauchy principal value*. One also finds that, as x_2 approaches ξ, $I_s(x_2,\xi)$ is singular for each ξ in $\mathbb{R}$ in a manner entirely analogous to that displayed above. On the other hand, an examination of (3.2.6) reveals that the behavior of $K_s(x_2,\xi)$ as either ξ approaches a fixed x_2 in $\mathbb{R}$ or as x_2 approaches a fixed ξ in

$\mathbb{R}$ is regular; in fact, since s is an element of $\mathcal{A}$, the limits

$$\lim_{\xi \to x_2} K_s(x_2, \xi) = -\frac{s''(x_2)}{2(1+s'(x_2)^2)} \quad \forall x_2 \in \mathbb{R},$$

and

$$\lim_{x_2 \to \xi} K_s(x_2, \xi) = -\frac{s''(\xi)}{2(1+s'(\xi)^2)} \quad \forall \xi \in \mathbb{R},$$

both exist and are finite.

Recall, now, that the function ϕ which appears in the second term on the right hand side of (3.2.6) is unknown. The jump condition $(3.1.17)_2$ serves, therefore, as a device by which this function can be determined. An appropriate substitution of (3.2.6) into $(3.1.17)_2$ yields—after collecting terms and dropping a non-vanishing common factor—the following equation:

$$(\mu_1+\mu_2)\phi + \frac{\mu_1-\mu_2}{\pi\sqrt{1+(s')^2}} \int_{-\infty}^{+\infty} K_s(\cdot,\xi)\phi(\xi)\sqrt{1+s'(\xi)^2}\,d\xi$$
$$= \frac{\mu_1-\mu_2}{\pi\sqrt{1+(s')^2}} \fint_{-\infty}^{+\infty} I_s(\cdot,\xi)s'(\xi)\,d\xi \quad \text{on} \quad \mathbb{R}. \tag{3.2.9}$$

Observe that (3.2.9) constitutes, for each fixed s contained in $\mathcal{A}$, a linear integral equation to be solved for ϕ on $\mathbb{R}$. The integral equation in (3.2.9) can be simplified by making a few modest substitutions; toward this end define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x_2) = \phi(x_2)\sqrt{1+s'(x_2)^2} \quad \forall x_2 \in \mathbb{R}, \tag{3.2.10}$$

and introduce a real constant λ through the relation

$$\lambda = \frac{\mu_1-\mu_2}{\pi(\mu_1+\mu_2)}. \tag{3.2.11}$$

Recall from the definition of τ_p that the moduli μ_1 and μ_2 satisfy $0 < \mu_2 < \mu_1$; λ must, consequently, lie strictly between 0 and $1/\pi$. Continuing with the simplification of (3.2.9), multiply and then divide both sides of the integral equation by

$\sqrt{1+(s')^2}$ and $(\mu_1+\mu_2)$, respectively, to obtain, with the aid of the definitions (3.2.10) and (3.2.11) the following alternative to (3.2.9):

$$\varphi + \lambda \int_{-\infty}^{+\infty} K_s(\cdot,\xi)\varphi(\xi)\,d\xi = \lambda \fint_{-\infty}^{+\infty} I_s(\cdot,\xi)s'(\xi)\,d\xi \quad \text{on} \quad \mathbb{R}. \tag{3.2.12}$$

For the purpose of facilitating the forthcoming discussion introduce, for each function s contained in $\mathcal{A}$, an operator $\mathcal{M}_s$ such that, for each function ψ the function $\mathcal{M}_s\psi$ is given by

$$\mathcal{M}_s\psi = \int_{-\infty}^{+\infty} K_s(\cdot,\xi)\psi(\xi)\,d\xi \quad \text{on} \quad \mathbb{R}. \tag{3.2.13}$$

In addition, let a function $f_s : \mathbb{R} \to \mathbb{R}$ be defined for each s in $\mathcal{A}$ via

$$f_s = \fint_{-\infty}^{+\infty} I_s(\cdot,\xi)s'(\xi)\,d\xi \quad \text{on} \quad \mathbb{R}. \tag{3.2.14}$$

With the aid of (3.2.13) and (3.2.14), (3.2.12) can be recast to read

$$\varphi + \lambda\mathcal{M}_s\varphi = \lambda f_s \quad \text{on} \quad \mathbb{R}. \tag{3.2.15}$$

Evidently a solution φ to (3.2.15) provides, through (3.2.10), a solution to (3.2.9). However, it is also clear from (3.2.1)–(3.2.3) and (3.2.10) that, given φ, v can be obtained directly in the form

$$v(x_1,x_2) = \frac{\gamma_l - \gamma_r}{2\pi} \int_{-\infty}^{+\infty} \left(G_1^s(x_1,x_2,\xi)\varphi(\xi) + G_2^s(x_1,x_2,\xi)s(\xi)\right)d\xi$$
$$\forall (x_1,x_2) \in \mathbb{R}^2 \setminus \mathcal{C}_s, \tag{3.2.16}$$

which obviates the need to consider ϕ. Hence, for each s in $\mathcal{A}$ the task of constructing a solution v to the corresponding reduced boundary value problem

(3.1.17) is altered, via potential theory, to one of constructing a solution to the corresponding integral equation (3.2.15). The task of the next section is to determine a set of conditions upon s—in addition to requiring that it to be an element of $\mathcal{A}$—which are sufficient to guarantee the existence of a solution to (3.2.15).

3.3. Analysis of the integral equation. Suppose that s is contained in $\mathcal{A}$ and consider the kernel K_s associated with it by (3.2.8). Observe that the stipulated smoothness of s implies that K_s is a continuous function on $\mathbb{R}^2$. Moreover, since

$$|K_s(x_2,\xi)| \leq \frac{|s(x_2) - s(\xi) - (x_2 - \xi)s'(x_2)|}{(x_2 - \xi)^2} \quad \forall (x_2,\xi) \in \mathbb{R}^2; \tag{3.3.1}$$

the boundedness of s'' on $\mathbb{R}$ and Taylor's theorem imply the following global estimate for the modulus of K_s:

$$|K_s(x_2,\xi)| \leq \frac{1}{2} \sup_{\zeta \in \mathbb{R}} |s''(\zeta)| = \frac{1}{2} \|s''\|_{L^\infty(\mathbb{R})} \quad \forall (x_2,\xi) \in \mathbb{R}^2. \tag{3.3.2}$$

Hence, the kernel K_s corresponding to any s in $\mathcal{A}$ is continuous and bounded on $\mathbb{R}$; furthermore, the bound is given explicitly in terms of a functional of s—the L^∞ norm of s'' over $\mathbb{R}$. If the integral equation held over a compact domain then the bound (3.3.2) would lead, for each fixed λ in $(0, 1/\pi)$, to sufficient conditions in terms of the size of $\|s''\|_{L^\infty(\mathbb{R})}$ which would allow the construction of a unique solution to the integral equation via a uniformly convergent *Neumann series*. Since the integral equation in (3.2.15) holds over $\mathbb{R}$, it will be convenient to determine conditions on functionals of s other than its L^∞ norm which are sufficient to guarantee an analogous result. Toward this end consider the Neumann series for this integral equation. This series is readily obtained via the method of successive substitutions and is given by $\Phi : \mathbb{R} \to \mathbb{R}$ as defined below:

$$\Phi = \lambda \sum_{n=0}^{\infty} (-\lambda)^n \mathcal{M}_s^n f_s \quad \text{on} \quad \mathbb{R}. \tag{3.3.3}$$

Observe that, with the aid of (3.2.13) and a formal interchange of summation and integration, Φ satisfies

$$\Phi + \lambda \mathcal{M}_s \Phi = \lambda f_s \quad \text{on} \quad \mathbb{R}.$$

That is, provided the formal operations performed above can be justified, Φ furnishes a solution to the integral equation. If the Neumann series converges uniformly then this is certainly the case. Consider the following geometric series:

$$g = \lambda \, \| f_s \|_{L^2(\mathbb{R})} \sum_{n=0}^{\infty} (\lambda)^n \, \| K_s \|^n_{L^2(\mathbb{R}^2)} \, . \tag{3.3.4}$$

If K_s and f_s are elements of $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R})$, respectively, and the L^2 norm of K_s over $\mathbb{R}^2$ satisfies

$$\lambda \, \| K_s \|_{L^2(\mathbb{R}^2)} < 1 \tag{3.3.5}$$

then (3.3.4) will converge. Note that the Neumann series is majorized by the geometric series. Conditions sufficient to guarantee the convergence of (3.3.4) are, accordingly, sufficient to assure that the Neumann series converges uniformly on its domain of definition and, therefore, as alluded to above, that Φ supplies a solution to (3.2.15). Provided these sufficient conditions are in force, the operator $\mathcal{M}_s$ is, moreover, a *Fredholm integral operator* with domain and range $L^2(\mathbb{R})$. Hence, the *Fredholm alternative* holds and it can be shown that the solution Φ to the integral equation provided by the Neumann series is unique.[4]

At present the aforementioned sufficient conditions are only of value if there exist functions s in the set $\mathcal{A}$ defined by (3.1.6)–(3.1.8) for which they hold. It will now be demonstrated that the first two conditions are satisfied for every s in $\mathcal{A}$ and that the third holds for every s contained in the proper subset $\mathcal{I}$ of $\mathcal{A}$ defined by

$$\mathcal{I} = \{ s \in \mathcal{A} \,|\, \lambda \sqrt{\tfrac{2\pi}{3}} (\int_{-\infty}^{+\infty} |s'(x_2)| |s''(x_2)| \, dx_2)^{\frac{1}{2}} < 1 \}. \tag{3.3.6}$$

[4] See GARABEDIAN [14] for a discussion of Neumann series and the foregoing results.

First suppose that s is an element of $\mathcal{A}$. Show that the kernel K_s must consequentially be square integrable on its domain of definition. Let $k_s : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$k_s(x_2,\eta) = \frac{s(x_2) - s(x_2+\eta) + \eta s'(x_2)}{\eta^2} \quad \forall (x_2,\eta) \in \mathbb{R}^2. \tag{3.3.7}$$

Note, from (3.3.1), (3.3.7) that

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K_s^2(x_2,\xi)\,d\xi\,dx_2 &\le \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{|s(x_2) - s(\xi) - (x_2-\xi)s'(x_2)|^2}{(x_2-\xi)^4}\,d\xi\,dx_2 \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k_s^2(x_2,\eta)\,d\eta\,dx_2. \end{aligned} \tag{3.3.8}$$

Hence, to demonstrate that K_s is contained in $L^2(\mathbb{R}^2)$ it suffices to show that k_s as defined in (3.3.7) is square integrable on $\mathbb{R}^2$. Now, with a formal change in the order of integration and the use of Parseval's identity the far right-hand-side of (3.3.8) can be recast as

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k_s^2(x_2,\eta)\,d\eta\,dx_2 &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k_s^2(x_2,\eta)\,dx_2\,d\eta \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\mathcal{F}\{k_s\}(\omega,\eta)|^2\,d\omega\,d\eta. \end{aligned} \tag{3.3.9}$$

The function $\mathcal{F}\{k_s\}(\cdot,\eta) : \mathbb{R} \to \mathbb{C}$ which appears in (3.3.9) represents, for each η in $\mathbb{R}$, the Fourier transform of $k_s(\cdot,\eta)$. This is supplied by

$$\begin{aligned}\mathcal{F}\{k_s\}(\omega,\eta) &= \int_{-\infty}^{+\infty} k_s(x_2,\eta)e^{-i\omega x_2}\,dx_2 \\ &= \hat{s}(\omega)\frac{1+i\omega\eta - e^{i\omega\eta}}{\eta^2} \quad \forall(\omega,\eta) \in \mathbb{R}^2, \end{aligned} \tag{3.3.10}$$

where $\hat{s} : \mathbb{R} \to \mathbb{C}$, in turn, is the Fourier transform of s:

$$\hat{s}(\omega) = \int_{-\infty}^{+\infty} s(x_2) e^{-i\omega x_2}\, dx_2 \quad \forall \omega \in \mathbb{R}. \tag{3.3.11}$$

A formal change in the order of integration on the far right-hand-side of (3.3.9) yields

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k_s^2(x_2,\eta)\, d\eta\, dx_2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left|\mathcal{F}\{k_s\}(\omega,\eta)\right|^2 d\eta\, d\omega,$$

so that, with the aid of (3.3.10) and (3.3.11),

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k_s^2(x_2,\eta)\, d\eta\, dx_2 &= \int_{-\infty}^{+\infty} |\hat{s}(\omega)|^2 \Big(\int_{-\infty}^{+\infty} \frac{|1 + i\eta\omega - e^{i\eta\omega}|^2}{2\pi\, \eta^4}\, d\eta\Big)\, d\omega \\ &= \int_{-\infty}^{+\infty} |\omega|^3 |\hat{s}(\omega)|^2\, d\omega \int_{-\infty}^{+\infty} \frac{|1 + i\zeta - e^{i\zeta}|^2}{2\pi\, \zeta^4}\, d\zeta. \end{aligned} \tag{3.3.12}$$

Next, a straightforward application of contour integration yields the identity

$$\int_{-\infty}^{+\infty} \frac{|1 + i\zeta - e^{i\zeta}|^2}{2\pi\, \zeta^4}\, d\zeta = \frac{1}{3},$$

so that (3.3.12) implies that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k_s^2(x_2,\eta)\, d\eta\, dx_2 = \frac{1}{3}\int_{-\infty}^{+\infty} |\omega|^3 |\hat{s}(\omega)|^2\, d\omega. \tag{3.3.13}$$

Note that, provided the integral on the right-hand-side of (3.3.13) exists, the two formal changes in the order of integration performed above are justified by Fubini's theorem.[5] Now, by (3.3.12), elementary identities involving the Fourier

[5] See HALMOS [17] for a statement and proof of Fubini's theorem.

transform of first and second derivatives, and Parseval's identity, (3.3.13) gives

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k_s^2(x_2,\eta)\, d\eta\, dx_2 = \frac{2\pi}{3}\int_{-\infty}^{+\infty} |s'(x_2)||s''(x_2)|\, dx_2. \tag{3.3.14}$$

Hence, provided s is an element of $\mathcal{A}$ it is apparent from (3.3.8), (3.3.14) and the Cauchy-Schwarz inequality that the kernel K_s is square integrable on $\mathbb{R}^2$ and, moreover, that $\|K_s\|_{L^2(\mathbb{R}^2)}$ can be estimated as follows:

$$\|K_s\|^2_{L^2(\mathbb{R}^2)} \leq \frac{2\pi}{3}\int_{-\infty}^{+\infty} |s'(x_2)||s''(x_2)|\, dx_2 \leq \tfrac{2\pi}{3}\, \|s'\|_{L^2(\mathbb{R})}\, \|s''\|_{L^2(\mathbb{R})}. \tag{3.3.15}$$

Observe that while the membership of s in $\mathcal{A}$ is certainly sufficient to ensure that K_s is an element of $L^2(\mathbb{R}^2)$ it is not necessary. An application of Hölder's inequality to (3.3.14) shows, for instance, that in order for K_s to be square integrable on $\mathbb{R}^2$ it is sufficient to require that s' and s'' be elements of $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$, respectively, for some p in $[1,\infty)$ and conjugate exponent $q = p/(p-1)$. The choice $p = q = 2$ clearly leads to the ultimate estimate in (3.3.14). It is, moreover, clear that, provided s is in $\mathcal{A}$, the domain and range of the operator $\mathcal{M}_s$ introduced in (3.2.13) can both be taken as $L^2(\mathbb{R})$. An immediate consequence of this observation is that if ψ is in $L^2(\mathbb{R})$ then so also is $\mathcal{M}_s^n\psi$ for any natural number n.

Next, given that K_s is an element of $L^2(\mathbb{R}^2)$ for every function s in $\mathcal{A}$, consider the issue of proving that the forcing f_s is similarly square integrable on its domain of definition. Observe, first, that the singular behavior of I_s which appears in the definition (3.2.15) suggests that f_s can be linearly decomposed into a regular part and a Cauchy principal value part as follows:

$$\begin{aligned} f_s(x_2) &= \int_{-\infty}^{+\infty}\Big(I_s(x_2,\xi) - \frac{1}{x_2-\xi}\Big)s'(\xi)\, d\xi + \fint_{-\infty}^{+\infty}\frac{s'(\xi)\, d\xi}{x_2-\xi} \\ &= -\int_{-\infty}^{+\infty} K_s(x_2,\xi)\frac{s(x_2)-s(\xi)}{x_2-\xi}s'(\xi)\, d\xi + \fint_{-\infty}^{+\infty}\frac{s'(\xi)\, d\xi}{x_2-\xi} \quad \forall x_2 \in \mathbb{R}. \end{aligned} \tag{3.3.16}$$

It is now convenient to define functions $g_s : \mathbb{R} \to \mathbb{R}$ and $h_s : \mathbb{R} \to \mathbb{R}$ by

$$g_s(x_2) = -\int_{-\infty}^{+\infty} K_s(x_2,\xi)\frac{s(x_2)-s(\xi)}{x_2-\xi}s'(\xi)\,d\xi \quad \forall x_2 \in \mathbb{R}, \tag{3.3.17}$$

and

$$h_s(x_2) = \fint_{-\infty}^{+\infty} \frac{s'(\xi)\,d\xi}{x_2-\xi} \quad \forall x_2 \in \mathbb{R}, \tag{3.3.18}$$

respectively. Consider the term of the decomposition involving the function g_s. From the assumed smoothness of s, the difference quotient which appears in the integrand on the right hand side of (3.3.17) satisfies

$$\left|\frac{s(x_2)-s(\xi)}{x_2-\xi}\right| \le \|s'\|_{L^\infty(\mathbb{R})} \quad \forall (x_2,\xi) \in \mathbb{R}^2,$$

and, hence, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}\int_{-\infty}^{+\infty} g_s^2(x_2)\,dx_2 &\le \|s'\|^2_{L^\infty(\mathbb{R})} \int_{-\infty}^{+\infty}\Big(\int_{-\infty}^{+\infty} K_s(x_2,\xi)s'(\xi)\,d\xi\Big)^2\,dx_2 \\ &\le \|s'\|^2_{L^\infty(\mathbb{R})} \int_{-\infty}^{+\infty}\Big(\int_{-\infty}^{+\infty} K_s^2(x_2,\xi)\,d\xi\Big)\Big(\int_{-\infty}^{+\infty} |s'(\xi)|^2\,d\xi\Big)\,dx_2 \\ &\le \|s'\|^2_{L^\infty(\mathbb{R})}\, \|K_s\|^2_{L^2(\mathbb{R}^2)}\, \|s'\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{3.3.19}$$

Therefore, since s' is contained in $L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$, (3.3.19) and the bound (3.3.15) on $\|K_s\|_{L^2(\mathbb{R}^2)}$ guarantee that g_s is square integrable on $\mathbb{R}$ and, furthermore, deliver the estimate

$$\|g_s\|_{L^2(\mathbb{R})} \le \sqrt{\tfrac{2\pi}{3}}\, \|s'\|_{L^\infty(\mathbb{R})}\, \|s'\|^{\frac{3}{2}}_{L^2(\mathbb{R})}\, \|s''\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\,. \tag{3.3.20}$$

Next, observe from (3.3.18) that h_s is a scalar multiple of the *Hilbert transform* of s' and recall that the Hilbert transform maps the space $L^q(\mathbb{R})$ into itself for

q in $(1, \infty)$.[6] Consequently, the square integrability of s' on $\mathbb{R}$ shows that h_s is also an element of $L^2(\mathbb{R})$. Since the space of square integrable functions on $\mathbb{R}$ in linear it is clear that f_s, as the sum of two functions contained in $L^2(\mathbb{R})$, is itself an element of $L^2(\mathbb{R})$. Hence, under the assumption that s is a member of the set $\mathcal{A}$ the forcing f_s must necessarily be square integrable on $\mathbb{R}$. Note that—based on the last statement and earlier remarks pertaining to the domain and range of the integral operator $\mathcal{M}_s$—the function $\mathcal{M}_s^n f_s$ is contained in $L^2(\mathbb{R})$ for every non-negative integer n. Hence, if the Neumann series (3.3.3) is uniformly convergent then φ must also be square integrable on $\mathbb{R}$.

Up to this point it has been shown, as proposed above, that K_s and f_s are square integrable on their domains of definition for every s contained in $\mathcal{A} \supset \mathcal{I}$. Finally, it is readily apparent from the primary estimate of $\|K_s\|_{L^2(\mathbb{R}^2)}$ given in (3.3.15) that if s is an element of the set $\mathcal{I}$ introduced in (3.3.6) then inequality (3.3.5) must hold. Hence, (3.3.5) is satisfied for every element s of $\mathcal{I}$. To recapitulate, observe that if s is in $\mathcal{I}$ then there exists a unique solution to the corresponding integral equation (3.2.14) given by the appropriate Neumann series (3.3.3).

There may exist solutions to (3.2.15) which are not obtainable via the Neumann series construction. Since, however, the solution to the integral equation obtained via this construction is unique for each s in $\mathcal{I}$ it is apparent from the above discussion that, should there exist any solutions to (3.2.15) which can be aquired by alternate means, these must correspond to curves $\mathcal{C}_s$ described by functions s which do not belong to $\mathcal{I}$ (and may not even belong to $\mathcal{A}$). It is interesting to speculate on whether some of these solutions might correspond to states wherein the phase boundaries manifest large slopes and/or curvatures akin to those exhibited by the *fingers* found in studies of porous media, solidification, and crystal growth.

Prior to concluding this section a few comments regarding the uniqueness of

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[6] This fact is established in RIESZ [22].

the solution to the reduced boundary value problem are in order. Let s be an element of $\mathcal{I}$. Then, by the foregoing results, the related reduced boundary value problem has a solution given by the appropriate Neumann series (3.3.3). It is known that the solution to the integral equation which issues from the reduced boundary value problem is unique. The uniqueness of the solution to the reduced boundary value problem is, however, still in question. It will now be shown that the solution of the reduced boundary value problem is unique—just as with the globally elliptic pairwise homogeneous equilibrium states—up to an arbitrary additive constant. To see this suppose that $v_1 : \mathbb{R}^2 \backslash \mathcal{C}_s \to \mathbb{R}$ and $v_2 : \mathbb{R}^2 \backslash \mathcal{C}_s \to \mathbb{R}$ are both solutions to the reduced boundary value problem corresponding to s in $\mathcal{I}$; define $w : \mathbb{R}^2 \setminus \mathcal{C}_s \to \mathbb{R}$ by their difference $(v_1 - v_2)$ on $\mathbb{R}^2 \setminus \mathcal{C}_s$. Then, from (3.1.17), w clearly satisfies the following boundary value problem:

$$\begin{aligned} w_{,\alpha\alpha} &= 0 \quad \text{on} \quad \mathbb{R}^2 \setminus \mathcal{C}_s, \\ \mu_1 \frac{\partial w}{\partial n}\big(s(\cdot)+,\cdot\big) &= \mu_2 \frac{\partial w}{\partial n}\big(s(\cdot)-,\cdot\big) \quad \text{on} \quad \mathbb{R}, \\ w\big(s(\cdot)+,\cdot\big) &= w\big(s(\cdot)-,\cdot\big) \quad \text{on} \quad \mathbb{R}, \\ w_{,\alpha}(x_1,\cdot)\mathbf{e}_\alpha &= \mathrm{o}(1) \quad \text{as} \quad x_1 \to \pm\infty \quad \text{on} \quad \mathbb{R}. \end{aligned} \tag{3.3.21}$$

From (3.2.16) it is readily apparent that a solution to (3.3.21) is provided, modulo an arbitrary additive constant, by

$$w(x_1,x_2) = \frac{\gamma_l - \gamma_r}{2\pi} \int_{-\infty}^{+\infty} G_1^s(x_1,x_2,\xi)\psi(\xi)\,d\xi \quad \forall (x_1,x_2) \in \mathbb{R}^2. \tag{3.3.22}$$

where $G_1^s : (\mathbb{R}^2 \setminus \mathcal{C}_s) \times \mathbb{R} \to \mathbb{R}$ is given by $(3.2.3)_1$, and $\psi : \mathbb{R} \to \mathbb{R}$ satisfies

$$\psi + \lambda \mathcal{M}_s \psi = 0 \quad \text{on} \quad \mathbb{R}. \tag{3.3.23}$$

It is clear, based on the assumption that s is in $\mathcal{I}$, that λ cannot be an eigenvalue of the operator $\mathcal{M}_s$. Hence, (3.3.23) has only the zero solution. Now, since the representation (3.3.22) for w is modulo an arbitrary additive constant, the functions v_1 and v_2 can differ at most, as stated above, by a constant.

During the course of this section it has been shown that, for each function s contained in $\mathcal{I}$ there exists, up to an arbitrary additive constant, a unique solution to the reduced boundary value problem (3.1.17). This solution corresponds to a deformation involving a non-planar interface $\mathcal{Q}_s$. Inasmuch as the reduced phase segregation requirement (3.1.18) has not yet been applied it is still unclear whether any of the aforementioned solutions give rise to globally elliptic two-phase equilibrium states. The next section will, therefore, focus on characterizing a subset of $\mathcal{I}$ for which there exist solutions to the reduced boundary value problem augmented by the (reduced) constraints of phase segregation. If a function s belongs to this subset of $\mathcal{I}$ the interface $\mathcal{Q}_s$ will qualify as a phase boundary.

3.4. Implementation and satisfaction of the reduced phase segregation requirement. Let s be an element of $\mathcal{I}$ and suppose that φ and (up to an additive constant) v are the corresponding solutions to the integral equation (3.2.15) and the reduced boundary value problem (3.1.17). If v is to provide—through (3.1.16)—a solution u to the constrained boundary value problem its gradient must comply with the reduced strain constraints (3.1.18). Let $\kappa : \mathbb{R}^2 \setminus \mathcal{C}_s \to \bar{\mathbb{R}}_+$ denote the *reduced shear strain field* given by

$$\kappa = v_{,\alpha} v_{,\alpha} \quad \text{on} \quad \mathbb{R}^2 \setminus \mathcal{C}_s. \tag{3.4.1}$$

Certainly $|v_{,1}|$ must be less than or equal to κ on $\mathbb{R}^2 \setminus \mathcal{C}_s$; hence, if the reduced shear strain field complies with

$$\kappa^{\frac{1}{2}} < \min\{\underset{*}{\gamma} - \gamma_r, \gamma_l - \overset{*}{\gamma}\} \quad \text{on} \quad \mathbb{R}^2 \setminus \mathcal{C}_s, \tag{3.4.2}$$

then both of the inequalities which comprise the reduced phase segregation requirement (3.1.18) will be satisfied. Notice that the foregoing condition is sufficient but not necessary to ensure the segregation of phases. It may, consequently, lead to overly conservative restrictions. Despite the strong restrictions which may be imposed by enforcing (3.4.2) in lieu of (3.1.18), it will be demonstrated that there exists a non-empty subset of $\mathcal{I}$ each element of which gives rise to a soluble reduced constrained boundary value problem with a reduced shear strain field κ that allows their satisfaction.

The following simple calculation shows that κ is subharmonic on $\mathbb{R}^2 \setminus \mathcal{C}_s$:

$$\begin{aligned}\kappa_{,\alpha\alpha} &= (v_{,\beta}\, v_{,\beta})_{,\alpha\alpha} = 2(v_{,\alpha\beta}\, v_{,\beta})_{,\alpha} \\ &= 2(v_{,\alpha\beta}\, v_{,\alpha\beta} + v_{,\alpha\alpha\beta}\, v_{,\beta}) \\ &= 2v_{,\alpha\beta}\, v_{,\alpha\beta} \geq 0 \quad \text{on} \quad \mathbb{R}^2 \setminus \mathcal{C}_s.\end{aligned}$$

The subharmonicity of κ on $\mathbb{R}^2 \setminus \mathcal{C}_s$ implies, given the decay properties of the gradient of v embodied by $(3.1.17)_4$, that its maximum values on $\mathring{\mathcal{D}}_s^l$ and $\mathring{\mathcal{D}}_s^r$ must occur in the limits approaching the curve $\mathcal{C}_s$ from the high and low strain sides, respectively. Hence, in determining whether the reduced shear strain field κ satisfies (3.4.2) it is sufficient to analyze its limiting behavior on either side of the curve $\mathcal{C}_s$. A convenient approach to this is afforded by examining the limits of the normal and tangential derivatives of v on either side of $\mathcal{C}_s$. From (3.2.6), (3.2.11) and (3.2.12) it is evident that the limiting values of the normal derivative of the the reduced out-of-plane displacement field are given by

$$\frac{\partial v}{\partial n}(s(x_2)-, x_2) = -\frac{\gamma_l \varphi(x_2)}{\sqrt{1 + s'(x_2)^2}} \quad \forall x_2 \in \mathbb{R} \tag{3.4.3}$$

on the high strain side of $\mathcal{C}_s$, and

$$\frac{\partial v}{\partial n}(s(x_2)+, x_2) = -\frac{\gamma_r \varphi(x_2)}{\sqrt{1 + s'(x_2)^2}} \quad \forall x_2 \in \mathbb{R} \tag{3.4.4}$$

on the low strain side of $\mathcal{C}_s$.

Let $\mathbf{l} : \mathbb{R} \to \mathcal{N}$ designate the unit tangent vector to $\mathcal{Q}_s$ defined by

$$\mathbf{l}(x_2) = \frac{s'(x_2)\mathbf{e}_1 + \mathbf{e}_2}{\sqrt{1 + s'(x_2)^2}} \quad \forall x_2 \in \mathbb{R}. \tag{3.4.5}$$

Then a calculation very similar to that used in obtaining (3.2.4) yields

$$\begin{aligned}\frac{\partial v}{\partial l}(s(x_2)\pm, x_2) = \quad & \frac{\gamma_l - \gamma_r}{2\pi\sqrt{1+s'(x_2)^2}} \int_{-\infty}^{+\infty} K_s(x_2,\xi)s'(\xi)\,d\xi \\ & + \frac{\gamma_l - \gamma_r}{2\pi\sqrt{1+s'(x_2)^2}} \fint_{-\infty}^{+\infty} I_s(x_2,\xi)\varphi(\xi)\,d\xi \\ & \pm \frac{\gamma_l - \gamma_r}{2\sqrt{1+s'(x_2)^2}} s'(x_2) \quad \forall x_2 \in I\!R \end{aligned} \tag{3.4.6}$$

for the limits of the tangential derivative of v on either side of the curve $\mathcal{C}_s$. Here, as in (3.2.4), $I_s(x_2,\cdot)$ and $K_s(x_2,\cdot)$ are given, for each x_2 in $I\!R$, by (3.2.5).

Turn now to the estimation of (3.4.3), (3.4.4) and $(3.4.6)_\pm$. Consider the limits of the normal derivative first. The following pair of inequalities follow immediately from (3.4.3) and (3.4.4):

$$\begin{aligned} \left|\frac{\partial v}{\partial n}(s(x_2)-, x_2)\right| &\le \gamma_l\,|\varphi(x_2)| \le \gamma_l\,\|\varphi\|_{L^\infty(I\!R)} \quad \forall x_2 \in I\!R, \\ \left|\frac{\partial v}{\partial n}(s(x_2)+, x_2)\right| &\le \gamma_r\,|\varphi(x_2)| \le \gamma_r\,\|\varphi\|_{L^\infty(I\!R)} \quad \forall x_2 \in I\!R. \end{aligned} \tag{3.4.7}$$

Hence, in order to bound the limits of the normal derivative of the reduced displacement field on either side of $\mathcal{C}_s$ it is only necessary to estimate the L^∞ norm of φ over $I\!R$. In Appendix A it is shown that if $\psi : I\!R \to I\!R$ is an element of the set $\mathcal{V}$ defined in (3.1.7) then $\|\psi\|_{L^\infty(I\!R)}$ exists and can be bounded as follows:

$$\|\psi\|_{L^\infty(I\!R)} \le 2(\tfrac{\pi}{2})^{\frac{1}{4}}\,\|\psi\|^{\frac{1}{2}}_{L^2(I\!R)}\,\|\psi'\|^{\frac{1}{2}}_{L^2(I\!R)}\,. \tag{3.4.8}$$

Recall from Section 3.3 that if s is contained in $\mathcal{I}$ then φ is square integrable on $I\!R$. Therefore, if φ' exists and is square integrable on $I\!R$ the inequality displayed in (3.4.8) can be used—with ψ replaced by φ—to obtain an estimate for the L^∞ norm of φ over $I\!R$. Suppose, from now on, that s is three times continuously differentiable element of $\mathcal{I}$ with a square integrable third derivative on $I\!R$. It can be readily shown that this is sufficient to guarantee that φ' exists and is an element of $L^2(I\!R)$. In Appendix B it is demonstrated that the L^2 norms of φ and φ' over $I\!R$ can be estimated, respectively, by

$$\|\varphi\|_{L^2(I\!R)} \leq \frac{c_1(1 + \|s'\|_{L^2(I\!R)} \|s''\|_{L^2(I\!R)}) \|s'\|_{L^2(I\!R)}}{1 - \lambda\sqrt{\frac{2\pi}{3}} \|s'\|^{\frac{1}{2}}_{L^2(I\!R)} \|s''\|^{\frac{1}{2}}_{L^2(I\!R)}}, \tag{3.4.9}$$

and

$$\begin{aligned} \|\varphi'\|_{L^2(I\!R)} \leq c_2 \big[& \|s''\|_{L^2(I\!R)} + \|s'\|^{\frac{3}{2}}_{L^2(I\!R)} \|s''\|_{L^2(I\!R)} \|s'''\|^{\frac{1}{2}}_{L^2(I\!R)} \\ & + \|s'\|^2_{L^2(I\!R)} \|s''\|^{\frac{3}{2}}_{L^2(I\!R)} \|s'''\|^{\frac{1}{2}}_{L^2(I\!R)} \\ & + \|s''\|^{\frac{1}{2}}_{L^2(I\!R)} \|s'''\|^{\frac{1}{2}}_{L^2(I\!R)} \|\varphi\|_{L^2(I\!R)} \\ & + \|s'\|^{\frac{1}{2}}_{L^2(I\!R)} \|s''\|_{L^2(I\!R)} \|s'''\|^{\frac{1}{2}}_{L^2(I\!R)} \|\varphi\|_{L^2(I\!R)} \big]. \end{aligned} \tag{3.4.10}$$

The constants c_1 and c_2 which appear in (3.4.9) and (3.4.10) are positive real numbers entirely independent of s. Note that the denominator in (3.4.9) is strictly positive since s is an element of $\mathcal{I}$. With the aid of (3.4.9) the estimate (3.4.10) for $\|\varphi'\|_{L^2(I\!R)}$ can be expressed completely in terms of $\|s'\|_{L^2(I\!R)}$, $\|s''\|_{L^2(I\!R)}$ and $\|s'''\|_{L^2(I\!R)}$. Hence, (3.4.9), (3.4.10), (3.4.8) and (3.4.7) give estimates for the moduli of the limiting values of the normal component of the gradient of v on either side of $\mathcal{C}_s$ in terms of the L^2 norms of the first three derivatives of s over $I\!R$.

To provide an estimate for κ it ramains to obtain bounds on the limiting values of the tangential derivative of v. Toward this objective, introduce a function $\Lambda : I\!R \to \bar{I\!R}_+$ by

$$\Lambda(x_2) = \int_{-\infty}^{+\infty} K_s(x_2,\xi)s'(\xi)\,d\xi + ⨍_{-\infty}^{+\infty} I_s(x_2,\xi)\varphi(\xi)\,d\xi + \pi s'(x_2) \quad \forall x_2 \in I\!R. \tag{3.4.11}$$

Then it is clear from (3.4.6) and (3.4.11) that

$$\left|\frac{\partial v}{\partial l}\,(s(x_2)\pm, x_2)\right| \le \frac{\gamma_l - \gamma_r}{2\pi}|\Lambda(x_2)| \le \frac{\gamma_l - \gamma_r}{2\pi}\,\|\Lambda\|_{L^\infty(\mathbb{R})} \quad \forall x_2 \in \mathbb{R}. \qquad (3.4.12)$$

Under the current assumption that s''' exists and is square integrable on $\mathbb{R}$ it can be shown that both Λ and Λ' are elements of $L^2(\mathbb{R})$. Inequality (3.4.8) can, therefore, be applied with ψ replaced by Λ; this leads, through (3.4.12), to a bound on the limiting values of the tangential component of the gradient of v on either side of $\mathcal{C}_s$. Given the bounds (3.4.9) and (3.4.10) for $\|\varphi\|_{L^2(\mathbb{R})}$ and $\|\varphi'\|_{L^2(\mathbb{R})}$ it is straightforward to derive estimates for the L^2 norms of Λ and Λ' over $\mathbb{R}$ in the form

$$\begin{aligned}\|\Lambda\|_{L^2(\mathbb{R})} \le c_3\big[\,&\|s'\|_{L^2(\mathbb{R})} + \|s'\|^{\frac{3}{2}}_{L^2(\mathbb{R})}\,\|s''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} + \|\varphi\|_{L^2(\mathbb{R})} \\ &+ \|s'\|_{L^2(\mathbb{R})}\,\|s''\|_{L^2(\mathbb{R})}\,\|\varphi\|_{L^2(\mathbb{R})}\big],\end{aligned} \qquad (3.4.13)$$

and

$$\begin{aligned}\|\Lambda'\|_{L^2(\mathbb{R})} \le c_4\big[\,&\|s''\|_{L^2(\mathbb{R})} + \|s'\|_{L^2(\mathbb{R})}\,\|s''\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\,\|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \\ &+ \|s'\|^{\frac{3}{2}}_{L^2(\mathbb{R})}\,\|s''\|_{L^2(\mathbb{R})}\,\|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \\ &+ \|s'\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\,\|s''\|_{L^2(\mathbb{R})}\,\|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\,\|\varphi\|_{L^2(\mathbb{R})} \\ &+ \|s'\|^{\frac{3}{2}}_{L^2(\mathbb{R})}\,\|s''\|^{2}_{L^2(\mathbb{R})}\,\|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\,\|\varphi\|_{L^2(\mathbb{R})} \\ &+ \|\varphi\|_{L^2(\mathbb{R})}\big].\end{aligned} \qquad (3.4.14)$$

respectively. Here c_3 and c_4 are positive real numbers which, like c_1 and c_2, are independent of s. Note that the L^2 norm of φ over $\mathbb{R}$ appears in the estimates for both $\|\Lambda\|_{L^2(\mathbb{R})}$ and $\|\Lambda'\|_{L^2(\mathbb{R})}$. Hence, with the aid of (3.4.9) these can be rewritten solely in terms of the L^2 norms of s', s'' and s''' over $\mathbb{R}$. As with the normal component of the gradient of v, a bound on the limiting values of the tangential component of the gradient of v on either side of $\mathcal{C}_s$ is given, from

(3.4.13), (3.4.14), (3.4.11), (3.4.12), (3.4.9) and (3.4.8) in terms of the L^2 norms of the first three derivatives of s over $\mathbb{R}$.

From the preceeding discussion it is clear that κ can be made arbitrarily small by reducing the size of $\|s'\|_{L^2(\mathbb{R})}$, $\|s''\|_{L^2(\mathbb{R})}$ and $\|s'''\|_{L^2(\mathbb{R})}$. More specifically, (3.4.9), (3.4.10), (3.4.13) and (3.4.14) can be substituted appropriately into (3.4.8) to compute, using (3.4.7) and (3.4.12), an upper bound for κ on $\mathring{\mathcal{D}}_s^l \cup \mathring{\mathcal{D}}_s^r$ in the form

$$\kappa \leq \Gamma^2\left(\|s'\|_{L^2(\mathbb{R})}, \|s''\|_{L^2(\mathbb{R})}, \|s'''\|_{L^2(\mathbb{R})}\right) =: \Gamma_s^2. \tag{3.4.15}$$

Define a set of functions $\mathcal{J}$ by

$$\mathcal{J} = \{s \in \mathcal{A} \,|\, \Gamma_s < \min\{\underset{*}{\gamma} - \gamma_r, \gamma_l - \overset{*}{\gamma}\}\} \cap \mathcal{X}, \tag{3.4.16}$$

where $\mathcal{X}$ is given by

$$\mathcal{X} = \{\psi : \mathbb{R} \to \mathbb{R} \,|\, \psi \in C^3(\mathbb{R}), \psi^{(n)} \in L^2(\mathbb{R}), n = 0, 1, 2, 3\}. \tag{3.4.17}$$

Then, from (3.4.2), (3.4.15) and (3.4.16), provided s is an element of the set Π defined by

$$\Pi = \mathcal{I} \cap \mathcal{J}, \tag{3.4.18}$$

there exists a solution—unique up to an additive constant—to the associated constrained boundary value problem (3.1.17)–(3.1.18) with phase boundary $\mathcal{Q}_s$. This solution defines a globally elliptic inhomogeneous two-phase equilibrium state and, therefore, establishes the existence result sought after here. Note—from the definition of Π—that the approach delineated above provides a means by which an uncountably infinite number of such states can be constructed. It is significant that the loading conditions related at the outset of Section 3.1 give rise to not only a globally elliptic pairwise homogeneous equilibrium state but also an uncountably infinite number of globally elliptic inhomogeneous two-phase equilibria. This result clearly reflects the underlying non-linearity of the problem.

As remarked earlier, there may exist globally elliptic inhomogeneous two-phase equilibrium states which cannot be constructed via the approach taken above and, thus, do not correspond to phase boundaries in the set Π. Under relaxed smoothness assumptions on s there may, however, exist still other globally elliptic inhomogeneous two-phase equilibrium states which can be constructed via Neumann series. In particular, under such relaxed circumstances, it may be possible to demonstrate the existence of states wherein the associated phase boundaries exhibit geometrical irregularities such as corners or cusps (recall that the existence of equilibria involving cusped phase boundaries has been established by ROSAKIS [23] in his work involving a special anisotropic material).

3.5. An example. Given the results of Sections 3.3 and 3.4 it is illuminating to consider a particular class of functions in the set $\mathcal{A}$ defined by (3.1.6) and determine a subset of this class of functions which are also contained in the set Π defined by (3.4.18). Toward this end, let s is given by

$$s(x_2) = \frac{h}{1+(\frac{x_2}{\ell})^2} \quad \forall x_2 \in \mathbb{R}, \tag{3.5.1}$$

where h and ℓ are both positive constants. A representative graph of s is displayed in Figure 3. Note that s is clearly an infinitely differentiable element of the set $\mathcal{A}$ for all values of the parameters h and ℓ. Let the ratio of h to ℓ be denoted by ϵ. The kernel K_s associated with s must, as a consequence of the results of Section 3.3, be square integrable on $\mathbb{R}^2$. In fact, from (3.3.15) one finds, after a bit of calculation, that the L^2 norm of K_s over $\mathbb{R}^2$ can be bounded as follows:

$$\|K_s\|_{L^2(\mathbb{R}^2)} \le \|k_s\|_{L^2(\mathbb{R}^2)} = \frac{\pi}{2}\epsilon. \tag{3.5.2}$$

In the latter, k_s is as defined in (3.3.7). Hence, it is clear from (3.2.10) and (3.3.6) that if—for a given choice of the moduli μ_1 and μ_2 which define the elliptic phases of the three-phase material with shear stress response function

τ_p—the parameter ϵ satisfies

$$\epsilon < 2\frac{\mu_1 + \mu_2}{\mu_1 - \mu_2}, \tag{3.5.3}$$

then the function s introduced above will be an element of $\mathcal{I} \cap C^\infty(I\!R)$. Assume, from now on, that s as defined in (3.5.1) is such that (3.5.3) holds. Then the Neumann series (3.3.3) converges uniformly on $I\!R$ to a solution of the intergal equation. With the aid of the decomposition (3.3.16) of the forcing f_s, the solution of the integral equation can be expressed as

$$\varphi(x_2) = \lambda \sum_{n=0}^{\infty} (-\lambda)^n (\mathcal{M}_s^n h_s)(x_2) + \lambda \sum_{n=0}^{\infty} (-\lambda)^n (\mathcal{M}_s^n g_s)(x_2) \quad \forall x_2 \in I\!R, \tag{3.5.4}$$

where $\mathcal{M}_s$ is as defined in (3.2.13). Given τ_p and, hence, the moduli μ_1 and μ_2, it can be readily shown that, for every ϵ which satisfies (3.5.3), the following order relations hold for each non-negative integer n:

$$\begin{aligned} (\mathcal{M}_s^n h_s)(x_2) &= O(\epsilon^{2n+1}) \quad \forall x_2 \in I\!R, \\ (\mathcal{M}_s^n g_s)(x_2) &= O(\epsilon^{2n+2}) \quad \forall x_2 \in I\!R, \end{aligned} \tag{3.5.5}$$

Therefore, facilitated by (3.5.4) and (3.5.5), φ can be represented in the form

$$\varphi(x_2) = 2\lambda h \ell^2 \int_{-\infty}^{+\infty} \frac{\xi \, d\xi}{(l^2 + \xi^2)^2 (\xi - x_2)} + O(\epsilon^2) \quad \forall x_2 \in I\!R. \tag{3.5.6}$$

An application of contour integration yields

$$\int_{-\infty}^{+\infty} \frac{\xi \, d\xi}{(l^2 + \xi^2)^2 (\xi - x_2)} = \frac{\pi}{2\ell^3} \frac{1 - (\frac{x_2}{\ell})^2}{(1 + (\frac{x_2}{\ell})^2)^2} \quad \forall x_2 \in I\!R,$$

so that (3.5.6) becomes, with the aid of (3.2.11),

$$\varphi(x_2) = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \frac{1 - (\frac{x_2}{\ell})^2}{(1 + (\frac{x_2}{\ell})^2)^2} \epsilon + O(\epsilon^2) \quad \forall x_2 \in I\!R. \tag{3.5.7}$$

Observe that (3.5.7) and (3.5.1) can be used in (3.2.16) and (3.1.16) to construct, for each appropriate pair (h,ℓ), an approximate solution to the reduced boundary value problem (3.1.17). Now, substitution of (3.5.1) and (3.5.7) into (3.4.3) and (3.4.4) delivers the following formulae for the limiting values of the normal derivative of v on either side of $\mathcal{C}_s$:

$$\begin{aligned}\frac{\partial v}{\partial n}(s(x_2)-,x_2) &= -\frac{\mu_1(\gamma_l-\gamma_r)}{\mu_1+\mu_2}\frac{1-(\frac{x_2}{\ell})^2}{(1+(\frac{x_2}{\ell})^2)^2}\epsilon+O(\epsilon^2) \quad \forall x_2\in I\!R,\\ \frac{\partial v}{\partial n}(s(x_2)+,x_2) &= -\frac{\mu_2(\gamma_l-\gamma_r)}{\mu_1+\mu_2}\frac{1-(\frac{x_2}{\ell})^2}{(1+(\frac{x_2}{\ell})^2)^2}\epsilon+O(\epsilon^2) \quad \forall x_2\in I\!R.\end{aligned} \tag{3.5.8}$$

Similarly, substitution of (3.5.1) and (3.5.7) into (3.4.6) gives rise to the following expressions for the limiting values of the tangential derivative of v on either side of $\mathcal{C}_s$:

$$\begin{aligned}\frac{\partial v}{\partial l}(s(x_2)-,x_2) &= \frac{2\mu_1(\gamma_l-\gamma_r)}{\mu_1+\mu_2}\frac{\frac{x_2}{\ell}}{(1+(\frac{x_2}{\ell})^2)^2}\epsilon+O(\epsilon^2) \quad \forall x_2\in I\!R,\\ \frac{\partial v}{\partial l}(s(x_2)+,x_2) &= -\frac{2\mu_2(\gamma_l-\gamma_r)}{\mu_1+\mu_2}\frac{\frac{x_2}{\ell}}{(1+(\frac{x_2}{\ell})^2)^2}\epsilon+O(\epsilon^2) \quad \forall x_2\in I\!R.\end{aligned} \tag{3.5.9}$$

The expansions in (3.5.8) and (3.5.9) show the dependence on ϵ of the limiting values of the normal and tangential derivatives of v on either side of $\mathcal{C}_s$. They readily imply that, for the function s indicated in (3.5.1), the associated reduced shear strain field κ introduced via (3.4.1) satisfies

$$\kappa = O(\epsilon^2) \quad \text{on} \quad I\!R\setminus\mathcal{C}_s. \tag{3.5.10}$$

An immediate consequence of (3.5.10) is that if ϵ is made sufficiently small the reduced phase segregation requirement (3.1.18) will then be satisfied and the relevant function s will be contained in Π.

Note, alternatively, that the L^2 norms, over $I\!R$, of the first three derivatives of the function s defined in (3.5.1) can be computed directly to give

$$\|s'\|_{L^2(I\!R)} = \sqrt{\tfrac{\pi}{4}}\frac{h}{\ell^{\frac{1}{2}}}, \quad \|s''\|_{L^2(I\!R)} = \sqrt{\tfrac{3\pi}{4}}\frac{h}{\ell^{\frac{3}{2}}}, \quad \|s'''\|_{L^2(I\!R)} = \sqrt{\tfrac{45\pi}{8}}\frac{h}{\ell^{\frac{5}{2}}}.$$

If the foregoing are substituted in the estimates (3.4.9), (3.4.10), (3.4.13), and (3.4.14) then it is straightforward to show, with appropriate use of (3.4.7), that the quantity Γ_s defined in (3.4.15) is of order ϵ—which corroborates the asymptotic results obtained above. Hence, if the parameters h and ℓ which appear in the definition of s are chosen so that ϵ is sufficiently small, Γ_s will satisfy

$$\Gamma_s < \min\{\underset{*}{\gamma} - \gamma_r, \gamma_l - \overset{*}{\gamma}\}, \tag{3.5.11}$$

and s will be an element of Π.

In either case a class of phase boundaries $\mathcal{Q}_s$ for which $\epsilon = h/\ell$ is sufficiently small emerges from the class of functions s given by (3.5.1). The reduced out-of-plane displacement field corresponding to each such s, is by using (3.5.1) and (3.5.4)–(3.5.7) in (3.2.16), given approximately by

$$\begin{aligned} v(x_1, x_2) \sim \; & \frac{\gamma_l - \gamma_r}{2\pi} \frac{h x_2}{\ell} \int_{-\infty}^{+\infty} \frac{d\xi}{(1+\xi^2)((\frac{x_1}{\ell})^2 + (\frac{x_2}{\ell} - \xi)^2)} \\ & + \frac{\gamma_l - \gamma_r}{2\pi} h \int_{-\infty}^{+\infty} \frac{1-\xi^2}{(1+\xi^2)^2} \ln\sqrt{(\tfrac{x_1}{\ell})^2 + (\tfrac{x_2}{\ell} - \xi)^2}\, d\xi \\ & \qquad \forall (x_1, x_2) \in I\!R^2 \setminus \mathcal{C}_s. \end{aligned} \tag{3.5.12}$$

An approximation to the corresponding primitive out-of-plane displacement field u is then calculated easily by substituting (3.5.12) appropriately into (3.1.16).

4. STUDY OF PHASE BOUNDARY KINETICS AND STABILITY

This chapter relies on the concept of a quasistatic motion introduced in Section 2.3. Recall that, in addition to the shear stress response function τ_p, the constitutive characterization of the material at hand includes a kinetic response function $\tilde{V}$ which, in the setting of a quasistatic motion, dictates the dependence of the normal velocity of a particle located at a point on a phase boundary on the driving traction acting at that point. Given the distribution of driving traction on a particular phase boundary it is therefore—through the kinetic relation—possible to discuss the kinetics and stability of that phase boundary in slow motions. For illustrative purposes this is done below in the context of the specific class of phase boundaries studied in Section 3.5. A similar analysis could, in principle, be performed for any function s in Π.

In Section 4.1 the driving traction f which acts on such a phase boundary is derived. It is demonstrated that f is composed of the sum of an *ambient* term f_0 which corresponds to the constant driving traction which would act on a planar phase boundary corresponding to a suitable globally elliptic pairwise homogeneous equilibrium state and higher order terms which represent the increment to the driving traction resulting from the non-planarity of the surface $\mathcal{Q}_s$.

In Section 4.2 the ambient term f_0 and the most significant non-constant term in the driving traction f are used in conjunction with $\tilde{V}$ and the kinetic relation to address phase boundary kinetics and stability.

4.1. The driving traction acting on an arbitrary element of a specific class of phase boundaries. Let s be given by (3.5.1) with $\epsilon = h/\ell$ chosen so that (3.5.11) is fulfilled. As discussed in Section 3.5, $\mathcal{Q}_s$ is then a phase boundary. Consider the computation of the driving traction acting on $\mathcal{Q}_s$. The simple manner in which $\mathcal{Q}_s$ can be parameterized implies that—in the present context of antiplane shear—the expression for the driving traction provided in (2.4.6) can be written as a function of one variable. Furthermore, it can be shown without difficulty that, for the special three-phase material with shear stress response function τ_p, the driving traction f is given by

$$f = \frac{\mu_1 - \mu_2}{2}\left(u_{,\alpha}(s(\cdot)+,\cdot)u_{,\alpha}(s(\cdot)-,\cdot) - \overset{*}{\underset{*}{\gamma}}\gamma\right) \quad \text{on} \quad \mathbb{R}. \tag{4.1.1}$$

The definition of v supplied in (3.1.16) readily furnishes the following expressions for $u_{,\alpha}(s(\cdot)\pm,\cdot)\mathbf{e}_\alpha$ on $\mathbb{R}$:

$$\begin{aligned} u_{,\alpha}(s(\cdot)-,\cdot)\mathbf{e}_\alpha &= \gamma_l\mathbf{e}_1 + v_{,\alpha}(s(\cdot)-,\cdot)\mathbf{e}_\alpha \\ &= \gamma_l\mathbf{e}_1 + \frac{\partial v}{\partial n}(s(\cdot)-,\cdot)\mathbf{n} + \frac{\partial v}{\partial l}(s(\cdot)-,\cdot)\mathbf{l} \quad \text{on} \quad \mathbb{R}, \\ u_{,\alpha}(s(\cdot)+,\cdot)\mathbf{e}_\alpha &= \gamma_r\mathbf{e}_1 + v_{,\alpha}(s(\cdot)+,\cdot)\mathbf{e}_\alpha \\ &= \gamma_r\mathbf{e}_1 + \frac{\partial v}{\partial n}(s(\cdot)+,\cdot)\mathbf{n} + \frac{\partial v}{\partial l}(s(\cdot)+,\cdot)\mathbf{l} \quad \text{on} \quad \mathbb{R}. \end{aligned} \tag{4.1.2}$$

It is possible to show, from (3.5.1), (3.1.7) and (3.4.5), that the unit normal and tangent vectors to $\mathcal{Q}_s$ satisfy the following order relations:

$$\mathbf{n}\cdot\mathbf{e}_1 = 1 + O(\epsilon) \quad \text{on} \quad \mathbb{R}, \quad \mathbf{l}\cdot\mathbf{e}_1 = O(\epsilon) \quad \text{on} \quad \mathbb{R}. \tag{4.1.3}$$

Hence, (3.1.16), (4.1.2) and (4.1.3) yield

$$\begin{aligned} u_{,\alpha}(s(\cdot)+,\cdot)u_{,\alpha}(s(\cdot)-,\cdot) = \gamma_l\gamma_r + \gamma_l\frac{\partial v}{\partial n}(s(\cdot)+,\cdot) + \gamma_r\frac{\partial v}{\partial n}(s(\cdot)-,\cdot) \\ + O(\epsilon^2) \quad \text{on} \quad \mathbb{R}. \end{aligned} \tag{4.1.4}$$

Now, if (3.5.8) is inserted appropriately in (4.1.4) and the result is substituted into (4.1.1) the driving traction along the phase boundary $\mathcal{Q}_s$ can be expanded in powers of ϵ as follows

$$f(x_2) = f_0 - \nu\frac{1-(\frac{x_2}{\ell})^2}{(1+(\frac{x_2}{\ell})^2)^2}\,\epsilon + O(\epsilon^2) \quad \forall x_2 \in \mathbb{R}, \tag{4.1.5}$$

with the constants f_0 and ν given by

$$f_0 = \frac{\mu_1-\mu_2}{2}(\gamma_l\gamma_r - \overset{*}{\underset{*}{\gamma}}\gamma), \quad \nu = \frac{(\mu_1-\mu_2)^2}{(\mu_1+\mu_2)}\gamma_l\gamma_r. \tag{4.1.6}$$

f_0 is the ambient or *base* driving traction which would hold on a planar phase boundary associated with a globally elliptic pairwise homogeneous equilibrium state with displacement field (3.1.1). Observe that since

$$-\frac{1}{8} \leq \frac{1-(\frac{x_2}{\ell})^2}{(1+(\frac{x_2}{\ell})^2)^2} \leq 1 \quad \forall x_2 \in I\!R,$$

the coefficient of the $O(\epsilon)$ term in the expansion of f provides a bounded correction to the base term f_0. This term is, for small ϵ, the most significant contribution to f which results from the deviation in the geometry of $\mathcal{Q}_s$ from planar. Note that f_0 can take on any real value whereas ν must be positive. Furthermore, because the difference $(\mu_1 - \mu_2)$ is squared in $(4.1.6)_2$, the positivity of ν holds even if the affiliations of the moduli μ_1 and μ_2 are reversed so as to be associated with the high and low strain phases of the material at hand. From (4.1.5) it is apparent that the $O(\epsilon)$ contribution to the distribution of driving traction on the phase boundary $\mathcal{Q}_s$ under consideration is a complicated function of position.. See Figure 4 for the graph of the function corresponding to the $O(\epsilon)$ term in the expansion of f. The results of ABEYARATNE [1], imply that the equilibria at hand must, in general, constitute metastable states.

4.2. Kinetics and stability of an arbitrary element of a particular class of phase boundaries. Given the expansion (4.1.5) consider the issue of analyzing the kinetics and stability of an arbitrary element of the class of phase boundaries at hand. Suppose that $\tilde{V}$ is twice continuously differentiable on $I\!R$. Then, as noted in the remarks following (2.3.9), the admissibility of $\tilde{V}$ requires that $\tilde{V}(0) = 0$ and $\tilde{V}'(0) \geq 0$. Assume, for the purposes of this discussion, that f_0 is not a critical point of $\tilde{V}$; note that this requires, in particular, that if $f_0 = 0$ then $\tilde{V}'(0) > 0$. See Figure 5 for examples of graphs of monotone and non-monotone kinetic response functions.

Let $\mathcal{Q}_s^c$ and $\mathcal{Q}_s^f$ be those subsets of the phase boundary $\mathcal{Q}_s$ defined as follows:

$$\mathcal{Q}_s^c = \{\mathbf{x} \in \mathcal{Q}_s \,|\, x_2 \in (-\ell, \ell)\}, \quad \mathcal{Q}_s^f = \mathcal{Q}_s \setminus \bar{\mathcal{Q}}_s^c.$$

Observe, from (4.1.5), that $\mathcal{Q}_s^c$ and $\mathcal{Q}_s^f$ correspond to the portions of $\mathcal{Q}_s$ upon which the $O(\epsilon)$ correction to f_0 is negative and positive, respectively. Note, also, that $\mathcal{Q}_s^c$ is the subset of $\mathcal{Q}_s$ whose geometry deviates most significantly from planar—that is, rougly speaking, the major portion of the bump which is associated with the graph of the function s given by (3.5.1) corresponds to the image of $(-\ell, \ell)$ under s. See Figure 4.

From the assumed smoothness of $\tilde{V}$, (4.1.5) and Taylor's theorem the normal velocity at a point on the phase boundary is given by

$$V_n(\mathbf{x}) = \tilde{V}(f(x_2)) = \tilde{V}(f_0) - \nu \tilde{V}'(f_0) \frac{1 - (\frac{x_2}{\ell})^2}{(1 + (\frac{x_2}{\ell})^2)^2}\, \epsilon + O(\epsilon^2) \quad \forall \mathbf{x} \in \mathcal{Q}_s. \quad (4.2.1)$$

In determining the kinetic tendencies of $\mathcal{Q}_s$ it is now convenient to consider two cases. These are $f_0 = 0$ and $f_0 \neq 0$. Note that the base globally elliptic pairwise homogeneous equilibrium state is mechanically stable only in the first of these two cases.

Consider the case $f_0 = 0$. Then, since $\tilde{V}(0) = 0$, (4.2.1) imples that

$$V_n(\mathbf{x}) = -\nu \tilde{V}'(0) \frac{1 - (\frac{x_2}{\ell})^2}{(1 + (\frac{x_2}{\ell})^2)^2}\, \epsilon + O(\epsilon^2) \quad \forall \mathbf{x} \in \mathcal{Q}_s. \qquad (4.2.2)$$

Since $\tilde{V}'(0)$ and ν are positive, it is apparent from (4.2.2) and $(4.1.3)_1$ that, to most significant order in ϵ, all points on $\mathcal{Q}_s^c$ tend to move in the $-\mathbf{e}_1$ direction while all points on $\mathcal{Q}_s^f$ tend to move in the $\mathbf{e}_1$ direction. That is, if $f_0 = 0$ then the phase boundary displays a proclivity to become planar.

Now consider the case where $f_0 \neq 0$. Suppose, first, that f_0 is positive. As such the dominant contribution to the normal velocity is, at all points on $\mathcal{Q}_s$, in the $\mathbf{e}_1$ direction. Recall that f_0 is assumed not to be a critical point of $\tilde{V}$; hence, since $f_0 \neq 0$, $\tilde{V}'(f_0)$ can be either positive or negative. If $\tilde{V}'(0) > 0$ then, since $\nu > 0$, the normal velocity of points on $\mathcal{Q}_s^c$ and $\mathcal{Q}_s^f$ will decrease and increase, respectively, on top of the ambient value $\tilde{V}(f_0)$. This, as in the case where $f_0 = 0$, indicates a tendency for the phase boundary to straighten out. If, however,

$\tilde{V}'(f_0) < 0$ then, since $\nu > 0$, the exact opposite occurs—the normal velocity of points on Q_s^c and Q_s^f will add positive and negative increments, respectively, to the ambient value $\tilde{V}(f_0)$. The ***protruding*** part of the phase boundary, if $\tilde{V}'(f_0) < 0$, portrays a tendency to ***grow*** while the ***flat*** part ***lags*** behind. The subcase where f_0 is negative is yields a completely analogous result. That is, when $f_0 < 0$ the phase boundary shows a propensity to become planar or develop a larger protrusion depending upon whether $\tilde{V}'(f_0)$ is positive or negative, respectively.

The foregoing discussion shows that the kinetics of a phase boundary Q_s in the class at hand are, to first order in ϵ, stable or unstable depending upon whether the kinetic response function is locally increasing or decreasing at the ambient driving traction f_0. If the constitutive description of a three-phase material with shear stress response function τ_p also includes a monotone increasing kinetic response function such as that depicted in Figure 5a it is clear that phase boundaries of the class under consideration will always be stable. If, on the other hand, the constitutive description incrudes a non-monotone kinetic response function like that depicted in Figure 5b it is always possible to choose γ_l and γ_r so that the phase boundary is unstable. These results suggest that it may be reasonable to classify those three-phase materials with shear stress response function τ_p as ***kinetically stable*** and ***kinetically unstable*** depending on whether the kinetic response function $\tilde{V}$ is a monotone or non-monotone function of its argument. Such a classification is consistent with that found by FRIED [13] in a linear stability analysis of planar phase boundaries in arbitrary three-phase materials subjected to a class of perturbations which encompasses the set of phase boundaries Π determined in Chapter 3.

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APPENDICES

Appendix A. In this appendix inequality (3.4.8) is established for all functions ψ contained in the set $\mathcal{V}$ defined in (3.1.7). Let $\chi : \mathbb{R} \to \mathbb{R}$ be an element of $\mathcal{V}$ with compact support about the origin; suppose, further, that $\chi(0) = 1$. Then, if ψ is contained in $\mathcal{V}$, one has the following inequality:[7]

$$\|\psi\|_{L^\infty(\mathbb{R})} \leq \|\chi\|_{\mathcal{V}} \, \|\psi\|_{\mathcal{V}} \,, \tag{A.1}$$

where

$$\|\psi\|_{\mathcal{V}} = \|\psi\|_{L^2(\mathbb{R})} + \|\psi'\|_{L^2(\mathbb{R})} \,.$$

Hence, (A.1) shows that all elements ψ of $\mathcal{V}$ are bounded on $\mathbb{R}$. The limit

$$\lim_{x_2 \to -\infty} \psi(x_2) e^{-\frac{1}{2}(\frac{x_2}{\ell})^2} = 0 \tag{A.2}$$

must, consequentially, hold for every (without loss of generality) positive real number ℓ and every function ψ in $\mathcal{V}$. Evidently, then, such a function ψ can be expressed as follows:

$$\psi(x_2) = \int_{-\infty}^{x_2} \frac{\partial}{\partial \xi} \left(\psi(\xi) e^{-\frac{1}{2}(\frac{x_2 - \xi}{\ell})^2} \right) d\xi \quad \forall x_2 \in \mathbb{R}. \tag{A.3}$$

Thus, from (A.3) and the Cauchy-Schwarz inequality it is clear that

$$\begin{aligned} |\psi(x_2)| &\leq \int_{-\infty}^{+\infty} \frac{|x_2 - \xi|}{\ell^2} |\psi(\xi)| e^{-\frac{1}{2}(\frac{x_2 - \xi}{\ell})^2} \, d\xi + \int_{-\infty}^{+\infty} |\psi'(\xi)| e^{-\frac{1}{2}(\frac{x_2 - \xi}{\ell})^2} \, d\xi \\ &\leq \pi^{\frac{1}{4}} \left(\tfrac{1}{\sqrt{2\ell}} \|\psi\|_{L^2(\mathbb{R})} + \sqrt{\ell} \, \|\psi'\|_{L^2(\mathbb{R})} \right) \quad \forall (x_2, \ell) \in \mathbb{R} \times \mathbb{R}_+. \end{aligned} \tag{A.4}$$

It is then obvious from (A.4) that

$$\|\psi\|_{L^\infty(\mathbb{R})} \leq \pi^{\frac{1}{4}} \left(\tfrac{1}{\sqrt{2\ell}} \|\psi\|_{L^2(\mathbb{R})} + \sqrt{\ell} \, \|\psi'\|_{L^2(\mathbb{R})} \right) \quad \forall \ell \in \mathbb{R}_+. \tag{A.5}$$

[7] See AUBIN [9] for a demonstration of this fact.

Now, minimize (A.5) with respect to ℓ to obtain (3.4.8). Note that the constant $2(\frac{\pi}{2})^{\frac{1}{4}}$ in (3.4.8) may not be the sharpest possible one for an estimate of this type. That is, there may exist a function ψ in $\mathcal{V}$ more optimal than the Gaussian used in (A.3)–(A.5).

Appendix B. In this appendix the estimates (3.4.9) and (3.4.10) for $\|\varphi\|_{L^2(\mathbb{R})}$ and $\|\varphi'\|_{L^2(\mathbb{R})}$ are established. First consider (3.4.9). From the integral equation in (3.2.15), the Minkowski inequality and the Cauchy-Schwarz inequality it is clear that

$$\|\varphi\|_{L^2(\mathbb{R})} \leq \lambda \|K_s\|_{L^2(\mathbb{R}^2)} \|\varphi\|_{L^2(\mathbb{R})} + \lambda \|f_s\|_{L^2(\mathbb{R})} . \tag{B.1}$$

With the aid of the decomposition of f_s provided in (3.3.17), the bound (3.3.21), and the fact $\|h_s\|_{L^2(\mathbb{R})} = \pi \|s'\|_{L^2(\mathbb{R})}$ (B.1) implies that

$$\|\varphi\|_{L^2(\mathbb{R})} \leq \frac{\lambda(\pi + \|s'\|_{L^\infty(\mathbb{R})} \|K_s\|_{L^2(\mathbb{R}^2)}) \|s'\|_{L^2(\mathbb{R})}}{1 - \lambda \|K_s\|_{L^2(\mathbb{R}^2)}}. \tag{B.2}$$

Now, use (3.4.8) and (3.4.9) in (B.2) to give (3.4.9).

Next consider (3.4.10). Recall that in order to obtain an estimate for the L^2 norm of φ' over $\mathbb{R}$ it is sufficient to require that s be an element of $\mathcal{I} \cap \mathcal{X}$, where $\mathcal{X}$ is given by (3.4.17). Suppose that this is the case. Then it is permissible to differentiate the integral equation in (3.2.15) to obtain

$$\varphi' + \lambda \int_{-\infty}^{+\infty} \tilde{K}_s(\cdot, \xi)\varphi(\xi)\, d\xi = \lambda f_s' \quad \text{on} \quad \mathbb{R}, \tag{B.3}$$

where

$$\tilde{K}_s(x_2, \xi) = 2K_s^2(x_2, \xi)\frac{s(x_2) - s(\xi)}{x_2 - \xi} + 2L_s(x_2, \xi) \quad \forall (x_2, \xi) \in \mathbb{R}^2, \tag{B.4}$$

and

$$L_s(x_2,\xi) = \frac{s(x_2) - s(\xi) - (x_2-\xi)s'(x_2) + \frac{1}{2}(x_2-\xi)^2 s''(x_2)}{(x_2-\xi)((s(x_2)-s(\xi))^2 + (x_2-\xi)^2)}$$

$$\forall (x_2,\xi) \in \mathbb{R}^2. \tag{B.5}$$

Clearly, with the aid of the Cauchy-Schwarz and Minkowski inequalities, (B.3) implies the following estimate for $\|\varphi'\|_{L^2(\mathbb{R})}$:

$$\|\varphi'\|_{L^2(\mathbb{R})} \leq \lambda \|\tilde{K}_s\|_{L^2(\mathbb{R}^2)} \|\varphi\|_{L^2(\mathbb{R})} + \lambda \|f_s'\|_{L^2(\mathbb{R})}. \tag{B.6}$$

So, given bounds for $\|\tilde{K}_s\|_{L^2(\mathbb{R}^2)}$ and $\|f_s'\|_{L^2(\mathbb{R})}$ in terms of the L^2 norms, over $\mathbb{R}$, of the first three derivatives of s, (B.6) will provide, in conjunction with (3.4.9), an estimate for $\|\varphi'\|_{L^2(\mathbb{R})}$. A bound for the L^2 norm of L_s over $\mathbb{R}^2$ can be obtained in exactly the same manner as that established for K_s in Section 3.3. This bound is

$$\|L_s\|_{L^2(\mathbb{R}^2)} \leq \frac{\sqrt{\pi}}{6} \|s''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})}. \tag{B.7}$$

Now, from (B.4), (B.5), (B.7), (3.3.2) the Cauchy-Schwarz inequality and the Minkowski inequality one obtains the following estimate for $\|\tilde{K}_s\|_{L^2(\mathbb{R}^2)}$:

$$\|\tilde{K}_s\|_{L^2(\mathbb{R}^2)} \leq \sqrt{\tfrac{2\pi}{3}} \|s'\|_{L^\infty(\mathbb{R})} \|s''\|_{L^\infty(\mathbb{R})} \|s'\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \|s''\|^{\frac{1}{2}}_{L^\infty(\mathbb{R})}$$
$$+ \tfrac{\sqrt{\pi}}{6} \|s''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})}. \tag{B.8}$$

It is also clear that

$$\|f_s'\|_{L^2(\mathbb{R})} \leq \pi \|s''\|_{L^2(\mathbb{R})} + \|s'\|_{L^\infty(\mathbb{R})} \|\tilde{K}_s\|_{L^2(\mathbb{R}^2)} \|s'\|_{L^2(\mathbb{R})}$$
$$+ \tfrac{1}{2} \|s''\|_{L^\infty(\mathbb{R})} \|K_s\|_{L^2(\mathbb{R}^2)} \|s'\|_{L^2(\mathbb{R})}. \tag{B.9}$$

Using (3.4.8), inequalities (B.8) and (B.9) become

$$\|\tilde{K}_s\|_{L^2(\mathbb{R}^2)} \leq \tfrac{4\pi}{\sqrt{3}} \|s'\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \|s''\|_{L^2(\mathbb{R})} \|s'''\|^{\frac{1}{2}}_{L^\infty(\mathbb{R})}$$
$$+ \tfrac{\sqrt{\pi}}{3} \|s''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})}, \tag{B.10}$$

and

$$\|f_s'\|_{L^2(\mathbb{R})} \leq \pi \|s''\|_{L^2(\mathbb{R})} + 2(\tfrac{\pi}{2})^{\frac{1}{4}} \|s'\|^{\frac{3}{2}}_{L^2(\mathbb{R})} \|s''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \|\tilde{K}_s\|_{L^2(\mathbb{R}^2)}$$
$$+ (\tfrac{\pi}{2})^{\frac{1}{4}} \|s'\|_{L^2(\mathbb{R})} \|s''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \|s'''\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \|K_s\|_{L^2(\mathbb{R}^2)}, \qquad \text{(B.11)}$$

respectively. Combining (B.6), (B.10), (B.11) and (3.3.15) leads, after a bit of algebra, to the desired estimate (3.4.10).

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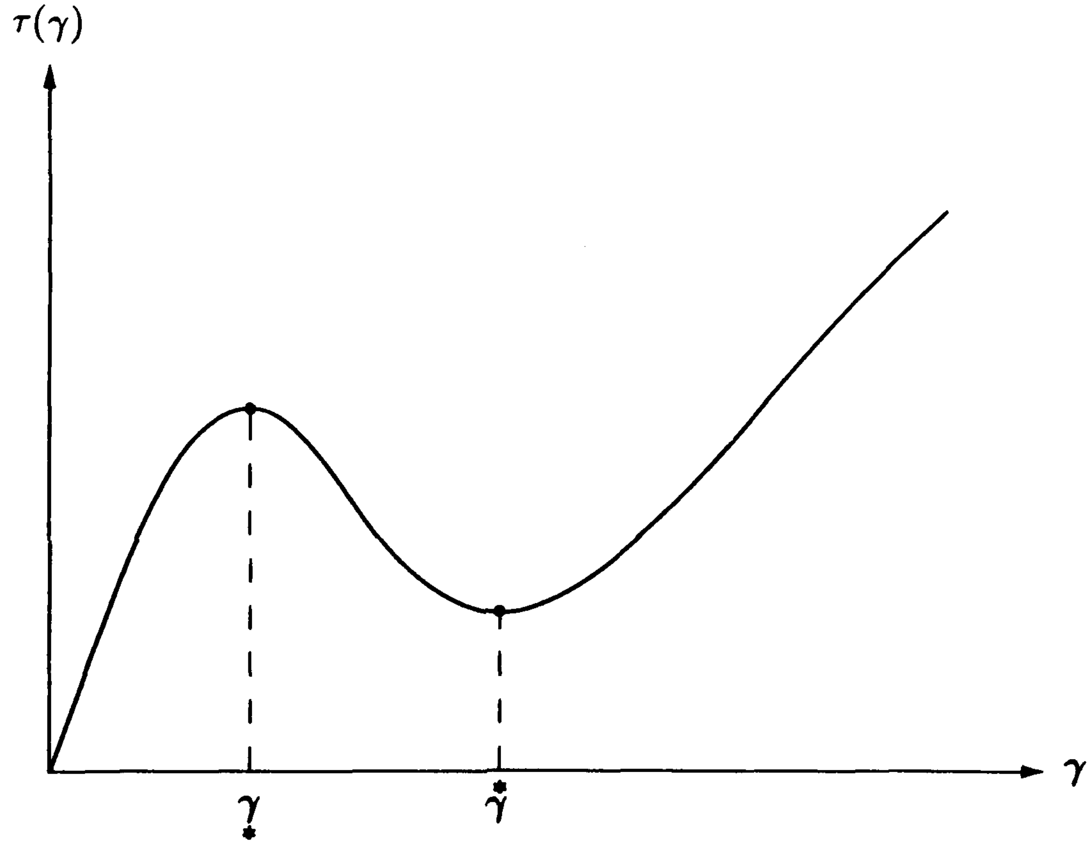

Figure 1: Graph of the shear stress response function τ.

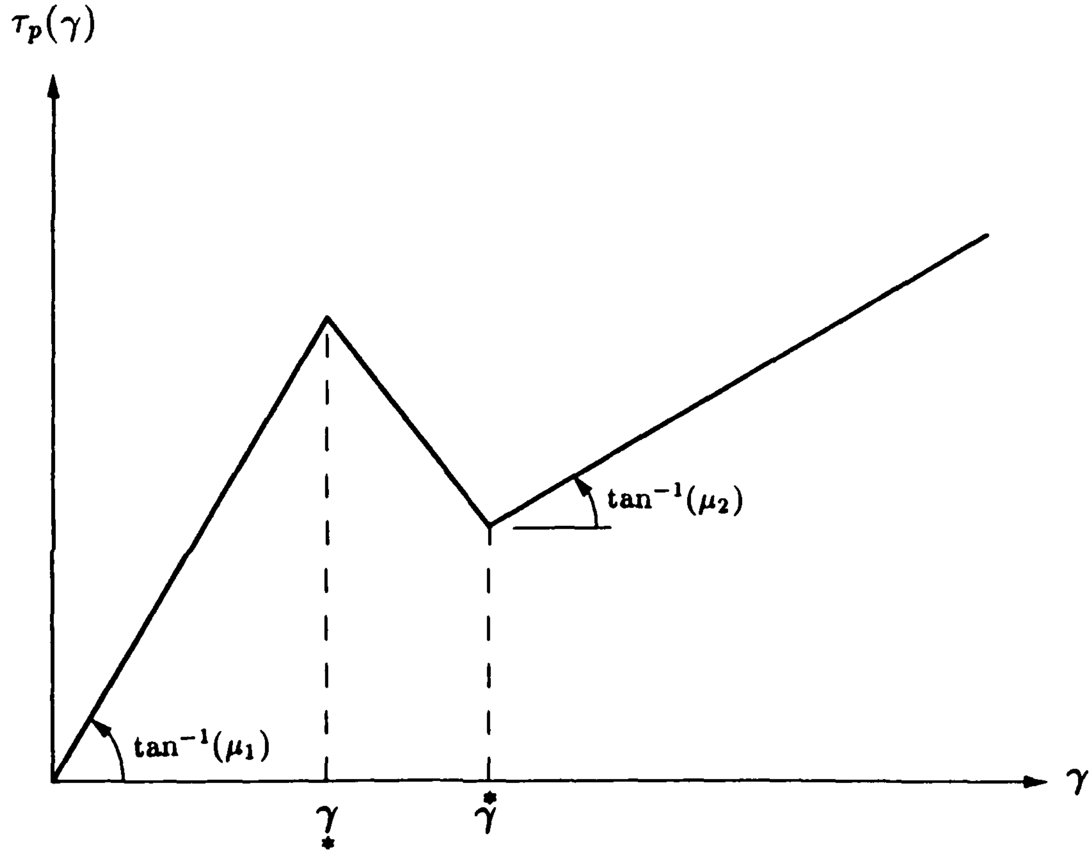

Figure 2: Graph of the shear stress response function τ_p.

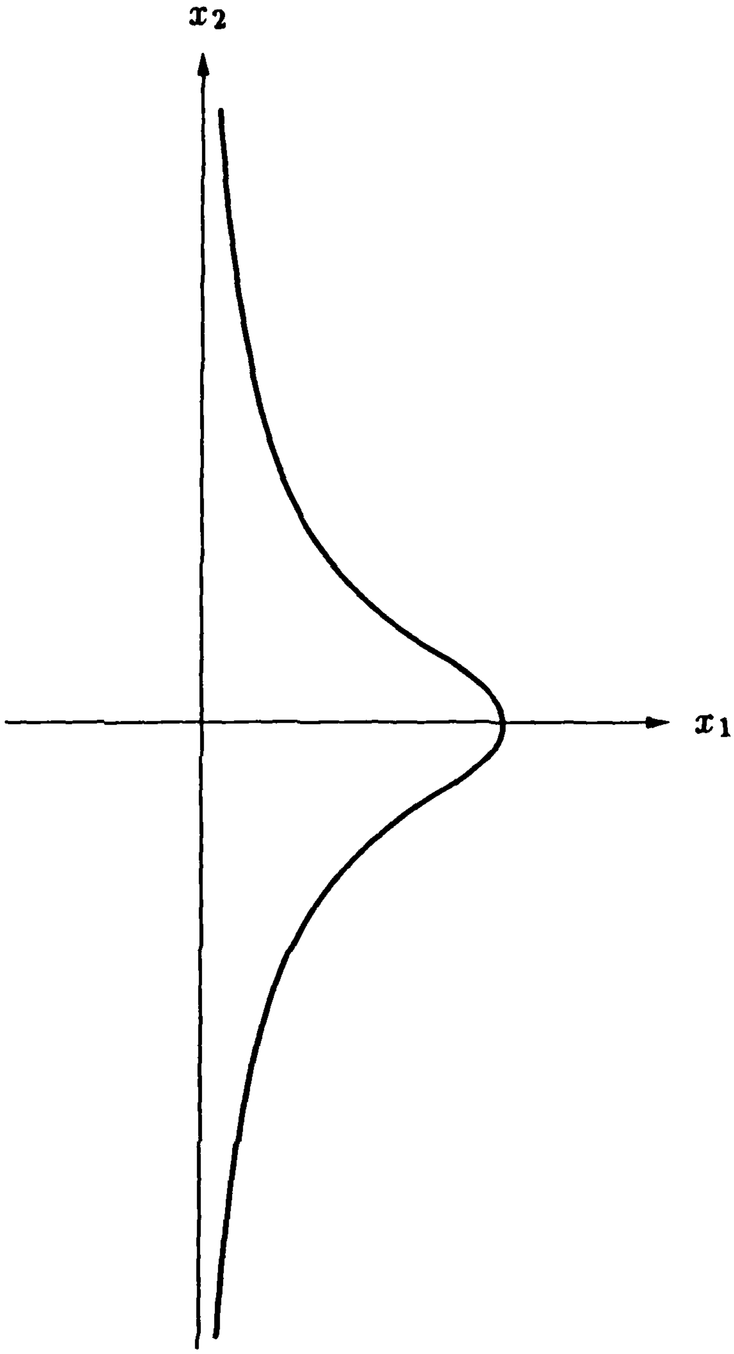

Figure 3: Graph of s.

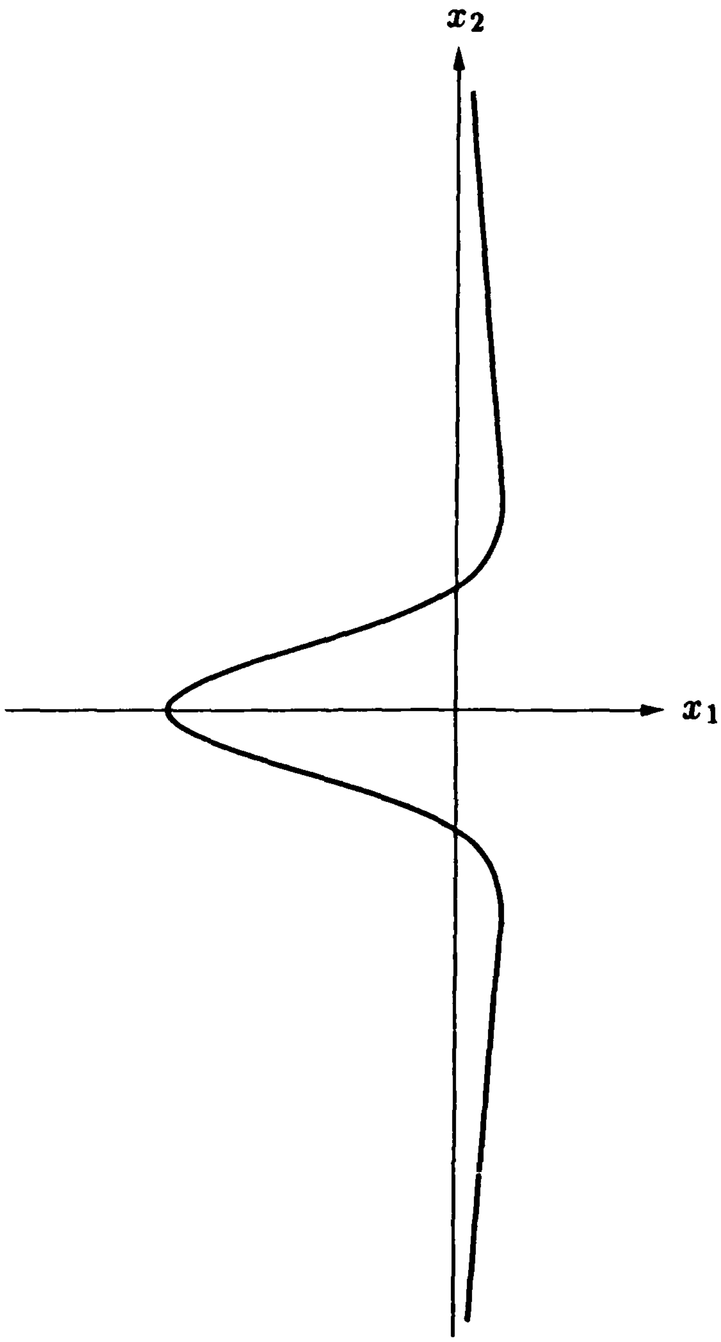

Figure 4: Graph of the $O(\frac{h}{l})$ correction to the driving traction.

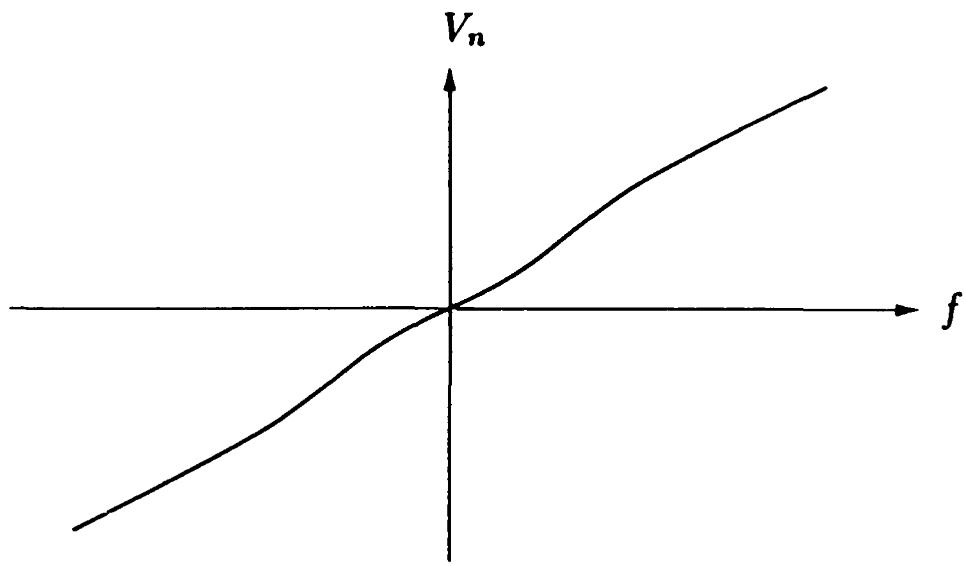

Figure 5a: Graph of a monotone increasing admissible kinetic response function.

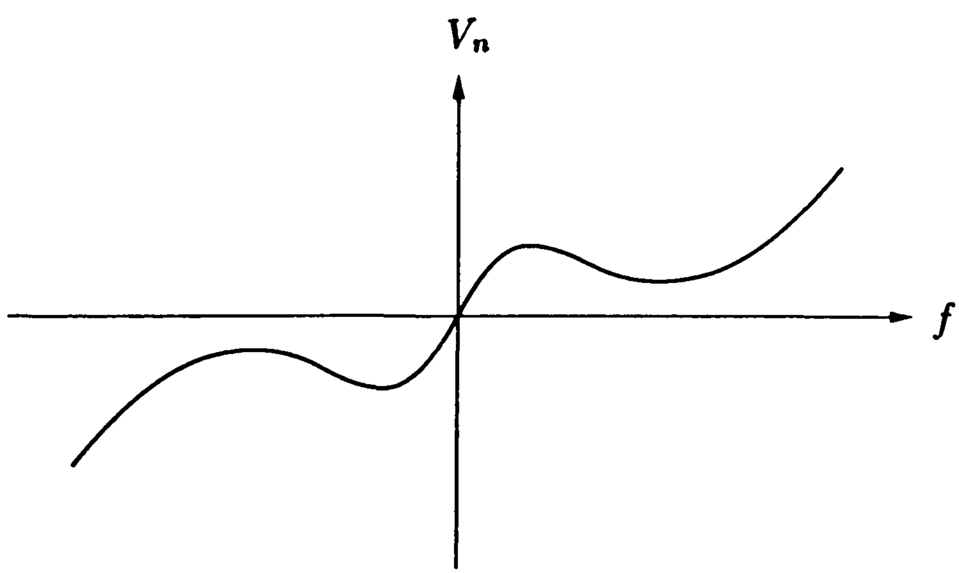

Figure 5b: Graph of a non-monotone admissible kinetic response function.

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This work focuses on the construction of equilibrated two-phase antiplane shear deformations of a non-elliptic isotropic and incompressible hyperelastic material. It is shown that this material can sustain metastable two-phase equilibria which are neither piecewise homogeneous nor axisymmetric, but, rather, involve non-planar interfaces which completely segregate inhomogeneously deformed material in distinct elliptic phases. These results are obtained by studying a constrained boundary value problem involving an interface across which the deformation gradient jumps. The boundary value problem is recast as an integral equation and conditions on the interface sufficient to guarantee the existence of a solution to this equation are obtained. The constraints, which enforce the segregation of material in the two elliptic phases are then studied. Sufficient conditions for their satisfaction are also secured. These involve additional restrictions on the interface across which the deformation gradient jumps--which, with all restrictions satisfied, constitutes a phase boundary. An uncountably infinite number of such phase boundaries are shown to exist. It is demonstrate

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